Excerpts (Chapter 6 - "Proof" and Chapter 6 Teacher Materials) from

Discovering the Art of Mathematics - Student Toolbox

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CHAPTER 6

Proof

Proof is an idol before which the mathematician tortures himself.

Sir Arthur Eddington (; -)

A elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline (; -)

A good proof is one that makes us wiser.

Yu. I. Manin (; -)

Proof is fundamental to what defines mathematics. "Proof" is a word that is used commonly and our everyday usage of the term is similar to mathematicians'. However, when used in mathematical settings there are fundamental and deep distinctions from the collquial usage. Understanding these distinctions are critical to understanding what mathematics truly is. For it is these distinctions which differentiate mathematics from all other fields of knowledge.

In mathematics precise definitions of terms are critical for it is upon these unambiguous definitions that the results of mathematics are logically established via proof. Ironically, the nature of proof itself has been the source of deep philosophical debate in mathematics through much of its history. In The Mathematical Experience **Philip J. Davis** (; -) and **Reuben Hersch** (; -) paint a portrait of an ideal mathematician.

> [The ideal mathematician] rests his faith on rigorous proof; he believes that the difference between a correct proof and an incorrect one is an unmistakable and decisive difference. He can think of no condemnation more damning than to say of a student, "He doesn't even know what a proof is." Yet he is able to give no coherent explanation of what is meant by rigor, or what is required to make a proof rigorous. In his own work, the line between complete and incomplete proof is always somewhat fuzzy, and often controversial.

The inability of the ideal mathematician to articulate what is meant by proof is then reinforced in wonderful dialogues between our ideal mathematician and a public information officer, a student, a philosopher, and a classicist.

Since there is no simple mathematical definition of "proof" that provides the working understanding you will need, it will be approached here incrementally with examples, investigations, and contexts.

1. First Proofs: The Magic of Nines

Here you will work to discover proofs that revolve around the magic of the number 9.

1.1. Finger Man.

> We are all concerned about the future of American education. But as I tell my students, you do not enter the future – you create the future. The future is created through hard work.

Jaime Escalante (Bolivian Teacher; 1930 - 2010)

Jaime Escalante (Bolivian Teacher; 1930 - 2010) is a well-known and inspiring teacher of mathematics whose work in the East Los Angeles schools received significant attention, most notably in the movie Stand and Deliver. In one scene, Escalante shows a confrontational student how to be a "finger man", computing the multiples of 9 on his fingers.

1. Watch this clip to learn this *algorithm* for computing multiples of 9.[1]
2. Describe this algorithm completely, using precise language that will allow someone who has never seen it to compute multiples of 9 using the algorithm.
3. *Prove* that this algorithm correctly computes each of the first nine multiples of 9.

Chisenbop is a powerful, twentieth century finger-counting algorithm developed in Korea which for computing multiples of 9 is identical to the method used above. Useful finger-counting methods have existed in many cultures through much of history, as long ago as the Vedic cultures of India over 3,000 years ago.

FIGURE 1. Jaime Escalante

1.2. Casting Out Nines. In fact, the number 9 has many remarkable properties in our base-ten number system.

> Remainder by Nines Problem - Find an efficient way to determine what the remainder will be when a whole number is divided by 9

This is the type of problem that mathematicians typically deal with. There is no obvious answer, no straightforward way to work it out. Instead, it requires experimentation, data collection, intuition, the search for connections to other problems, insight, etc. All of this is active work. Please get to work on it.

A typical timeline for a problem like this is that it should take an hour or so of active, focussed work. But this work need not come all at once, it can happen over several different sittings - with breaks inbetween to reflect on the problem a bit less actively, to do something totally different so your subconscious can work on it, or just to take a break.

Note: You should *not* look for a solution from an external resource - book, Internet, or peer. Real mathematical experiences are about problem solving, constructing understanding, developing sense-making and practicing the creation of ideas. You will not have an authentic mathematical experience, nor will you be able to succeed in the remainder of the explorations in Discovering the Art of Mathematics if you rely on others for all of your significant work. You may not ever need to use this problem again, but strengthening your abilities to problem solve, reason, understand how/why things work, and make sense of new objects, ideas, and settings are critical to real educational growth.

[1] It appears at minutes 8:30 - 10:00 in the original movie and can often be found online by searching for videos with keyword "Stand and Deliver Finger Man."

4. Precisely describe how you went about solving this problem.
5. Precisely describe your solution to the Remainder by Nines problem. (Note: Your solution should efficiently enable you to determine the remainder when *any* whole number is divided by 9. If you cannot do this for every number, you only have a partial solution thus far.)
6. Are you confident that your solution is legitimate? Explain what it is that makes you confident.

At present, your result in Investigation **5** is a **conjecture**, a statement you believe to be true. The analogous term in the sciences is hypothesis. While you may be confident in your result, it needs to be proven. If proven successfully, your result will no longer be considered a conjecture, but rather a **theorem**, that is mathematical certainty which has been established through logical deduction and is therefore eternal.

In proving the algorithm for computing multiples of nine, you may have used a ***proof by exhaustion***. There are only 9 cases to check. If you check each one, you have proof that it works. Of course, that is not possible here as there are infinitely many numbers to check. Alternatively, you may have proven the algorithm ***inductively***. It works for the first multiple. Each time I want the next multiple of nine I can think of adding ten and subtracting one - which is exactly what I get from depressing the very next finger. This will work until I run out of fingers, proving that the second multiple must follow from the truth of the first, the third from the truth of the second, etc.

The essential component of either of these proofs, and of any other, is that it establishes the desired result with certainty. There is a precise, logical way to follow a line of reasoning from the *assumptions* to the *conclusion*.

7. Prove that your solution to the Remainder by Nines problem is correct.
8. There is a rule for determining whether a whole number is divisible by 9. You should be able to determine this rule from your solution to the Remainder by Nines problem. But also look it up. Describe the rule precisely.
9. Explain why this rule follows logically as a direct result of your solution to the Remainder by Nines problem.

When a mathematical Your result in Investigation **8** is called a ***corollary*** because it is a theorem which is established as an immediate logical consequence of your solution to the Remainder by Nines problem.

1.3. The Human Calculator. In video available online at `http://www.artofmathematics.org/???` one of the authors of this book is shown as a "Human Calculator", adding up a column of eight 6-digit numbers as fast as he can write down the answer. Watch this trick, see if you can discover anything that might help you understand how this trick was performed.

Here's another example of that same trick:

Student	395014
Mathemagician	604985
Student	759231
Mathemagician	240768
Student	545310
Mathemagician	454689
Student	489132
Mathemagician	200543
	3689672

10. Describe some observations you have made in trying to discover how this trick works.
11. Find some relationships between these observations that may help discover how this trick works.
12. Determine how the trick works.
13. Now you become the mathemagician, try this trick on several friends. How'd it go? Were they impressed?
14. Explain precisely what you have to do in each stage of the trick for it to work.
15. Under the conditions just described, prove that the trick will always work.

16. Do you have to use 6-digit numbers to do the trick? Does there have to be eight numbers chosen? If so, explain why. If not, explain how the trick can be ***generalized***.

2. Where Do Proofs Come From?

> But how do we do it [discover a proof]? Nobody really knows. You just try and fail and get frustrated and hope for inspiration. For me it's an adventure, a journey. I usually know more or less where I want to go, I just don't know how to get there. The only thing I *do* know is that I'm not going to get there without a lot of pain and frustration and crumpled-up paper Then, all of a sudden, in one breathless heart-stopping moment, the clouds part and you can finally *see*... The thing I want you especially to understand is this feeling of divine revelation. I feel that this structure was "out there" all along; I just couldn't see it. And now I can! This is really what keeps me in the math game - the chance that I might glimpse some kind of secret underlying truth, some sort of message from the gods.[2]
>
> **Paul Lockhart** (; -)

> Now where did this idea of mine come from? How did I know to draw that line? How does the painter know where to put his brush? Inspiration, experience, trial and error, dumb luck. That's the art of it, creating these beautiful little poems of thought, these sonnets of pure reason. There is something so wonderfully transformational about this art form. The relationship between the triangle and the rectangle was a mystery, and then that one little line made it obvious. I couldn't see, and then all of a sudden I could. Somehow, I was able to create a profound simple beauty out of nothing, and change myself in the process. Isn't that what art is all about?[3]
>
> **Paul Lockhart** (; -)

3. What Do Proof's Look Like?

3.1. An Existence Proof.

17. Westfield State University, the oldest coeducational teachers college in the country, has an enrollment of about 4,500 students. Prove that there are at least two students who share the same birthday.

If you know two students at Westfield State with the same birthday, such a proof is called an ***explicit proof***. If you prove the result without knowing explicit examples, this is called an ***existence proof***.

3.2. The Role of Examples. Examples play important roles in mathematics. An important part of mathematics is inductive - based on generalization of examples and patterns. But mathematical results are not considered valid until they have been proven in general.

In a proof by exhaustion, *all* possible examples are considered, so this provides proof.

One can find ***counter-examples***, examples that disprove a conjecture or statement. For example, the number 2 is a counter-example to the statement "all prime numbers are odd." This one example is all that is required to disprove the statement. The **Riemann hypothesis**, one of the most important problems in all of mathematics and one which currently has a 1$ million dollar prize for its solution, states that all non-trivial roots of the Riemann zeta function lie on the line $x = \frac{1}{2}$. If someone finds just *one* non-trivial root of the Riemann zeta function that is not on this line they will have disproven the Riemann hypothesis, becoming 1$ million richer and having shocked the mathematical community to its core.

[2] From A Mathematician's Lament, pp. 113-4.
[3] From A Mathematician's Lament, pp. 27.

One *cannot* do more with examples in the way of proof. There is no such thing as a ***proof by example*** where an incomplete number of examples are taken to form proof. For emphasis, consider the following "proof" which utilizes almost infinitely many examples:

Conjecture Every day is Friday.[4]

Proof Today, 10/18/2013 is a Friday. 10/11/2013 was a Friday. 10/4/2013 was a Friday... Moreover, 10/25/2013 will be a Friday. 11/1/2013 will be a Friday. 11/8/2013 will be a Friday... Based on these examples, moving forward and backward to the beginning and end of the universe (if such things exist) prove that every day is Friday. //

3.3. An Infinite Result. Consider the expression $3a + 5b$ where $a, b \geq 0$ represent whole numbers.

18. Choose whole number values for $a, b \geq 0$. For these values, evaluate the expression $3a + 5b$.
19. Repeat Investigation **18** for a different pair of values for a, b.
20. Repeat Investigation **18** for another dozen different pairs of values for a, b.
21. Is every positive whole number a possible as the result of evaluating $3a + 5b$, or are there some numbers that cannot be made from this expression?
22. Continue investigating until you feel comfortable making a conjecture which identifies exactly which, all infinitely many of them, positive whole numbers can be made by evaluating the expression $3a + 5b$ for $a, b \geq 0$ being whole numbers.
23. Prove your conjecture.

3.4. Pythagorean Theorem. The *Pythagorean theorem* is one of the most well-known results in all of mathematics. While it cears the name of Pythagoras, it was known to many other cultures. Here you will re-discovery a proof attributed to **Bhaskara** (Indian mathematician; -).

There are *many* proofs of the Pythagorean theorem. **Elisha Scott Lewis** (; -) wrote The Pythagorean Proposition which is a collection of 367 proofs of the Pythagorean theorem!

In 1867, when he was a U.S. Representative, future President **James A. Garfield** (American politician; -) created a proof of the Pythagorean theorem that was published in the *New England Journal of Education*.

24. State the Pythagorean theorem. Be sure to include any assumptions that the theorem requires.
25. The proof uses paper triangles that you should create as follows:
- Fold an $8\frac{1}{2}$ by 11 sheet of paper in half in one direction and then in half again in the opposite direction.
- From the corner which is not along any of the fold lines, measure some distance up along one edge and the same distance along the other edge.
- Draw a line between the two points you have just measured, creating a right, isoceles triangle.
- Cut along this line to create four congruent copies of your triangle.
- Label the legs and the hypotenuses as a, b, and c.

26. Can you arrange your four triangles so they form the interior of a square whose are is c^2?
27. Now rearrange your four triangles to form two smaller squares.
28. Explain how this proves the Pythagorean theorem for your triangle.
29. Will this same proof work for any right, isoceles triangle? Explain.

Now adapt this method for a general right triangle as follows.

30. Repeat the folding, cutting and labelling process used above to create four congruent, right triangles that are *not* isoceles.
31. Try to arrange your four triangles so they form the interior of a square whose area is c^2. You should discover a missing piece.
32. With the paper scraps left over from your initial construction, make a piece that enables you to fill in the missing component of your c by c square.

[4] This was written on Friday, 10/18/2013.

33. Now rearrange your triangles and addition piece to form two smaller squares.
34. In both of the previous steps you should justify how you know that the shapes you have created are indeed squares and have the appropriate dimensions.
35. Explain how this proves the Pythagorean theorem for your triangle.
36. Will this same proof work for any right triangle? Explain.

4. Role and Importance of Proofs

> Math is not about a collection of "truths" (however useful or interesting they may be). Math is about reason and understanding. We want to know *why*. And *not* for any practical purpose. Here's where the art has to happen. Observation and discovery are one thing, but *explanation* is quite another. What we need is a *proof*, a narrative of some kind that helps us to understand why this pattern is occurring. And the standards for proof in mathematics are pretty damn high. A mathematical proof should be an absolutely clear logical deduction, which, as I said before, needs not only to satisfy, but to satisfy beautifully. That is the goal of the mathematician: to explain in the simplest, most elegant and logically satisfying way possible. To make the mystery melt away and to reveal a simple, crystalline truth.[5]
>
> **Paul Lockhart** (; -)

Here is where inductive reasoning goes.

4.1. Galileo Sums.

$$
\begin{aligned}
1 &= 1\\
1+3 &= ?\\
1+3+5 &= ?\\
1+3+5+7 &= ?\\
\vdots &= \vdots\\
1+3+5+7+\ldots+\boxed{\text{algebraic expression}} &= \boxed{\text{algebraic expression}}\\
\vdots &= \vdots
\end{aligned}
$$

37. Using square tiles or blocks, or graph paper to mimic them, can you arrange one tile of one color and three tiles of another color into an appropriate shape to illustrate the validity of the first equation?
38. Using one square tile of one color, three of another color and five of yet another color, can you arrange these tiles into an appropriate shape to illustrate the validity of the second equation?
39. Show how this process can be continued to provide a *Proof Without Words* for the general result.

This example has critical historical significance. In his inclined plane experiments, **Galileo** (; -) showed successfully that the distance a free falling body falls in successive time intervals is in the ratio of $1:3:5:7:9\cdots$. Over equal time intervals, summing these distances shows that the total

[5]From A Mathematician's Lament, pp. 110-1.

distance is proportional to the square of the time.[6] I.e. this pattern is encoded in uniform motion at the most fundamental level.

4.2. Galileo Fractions.

$$\frac{1}{3} = \frac{1}{3}$$
$$\frac{1+3}{5+7} =?$$
$$\frac{1+3+5}{7+9+11} =?$$
$$\vdots = \vdots$$

40. Show how Figure 8 shows that $\frac{1+3}{5+7} = \frac{1}{3}$.
41. Generalized Figure 8 and your observation above to provide a Proof Without Words for Galileo fractions.[7]:

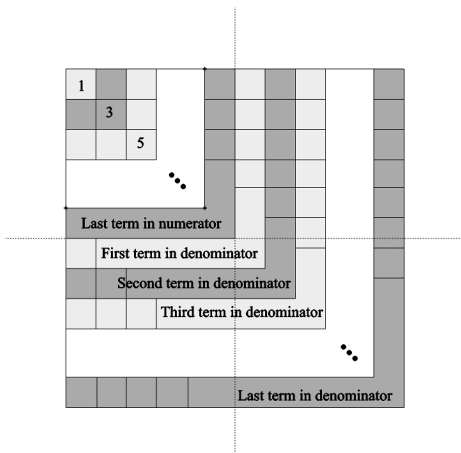


FIGURE 2. Proof Without Words: Galileo fractions.

4.3. Pennies and Paperclips. Pennies and Paperclips[8] is a two player game played on a board resembling a checkerboard. One player, "Penny", gets two pennies as her pieces. The other player, "Clip", gets a pile of paperclips as his pieces. Penny places her two pennies on any two different squares on the board. Once the pennies are placed, Clip attempts to cover the remainder of the board with paperclips - with each paperclip being required to cover two adjacent squares. Paperclips are not allowed to overlap. If the remainder of the board can be covered with paperclips

[6]See Chapter 1 - "Galileo and Why Things Move" from The Ten Most Beautiful Experiments by George Johnson for a secondary description, "Galileo's Experimental Confirmation of Horizontal Inertia" by Stillwell Drake, *Isis*, vol. 64, no. 3, pp. 290-305 for a primary source description, and Section 4.2 - "The Studies of Galileo" in Basic Calculus: From Archimedes to Newton to Its Role in Science by Alexander J. Hahn for an introductory physics/calclus level description.

[7]This approach to proving this result is from "Proof Without Words: Galileo's Ratios Revisited" by Alfino Flores and Hugh A. Sanders, *College Mathematics Journal*, vol. 36, no. 3, May 2005, p. 198.

[8]The history of this game is not know to the author. It appears in the 1994 version of Harold R. Jacobs' Mathematics: A Human Endeavor. In the 1970 version of this book the "mutilated checkerboard" appears instead. The latter is from the 1950's and its noteriety is due to Martin Gardner's *Scientific American* columns. It may well be that the translation of the checkerboard problem into a game is due to Jacobs.

then Clip is declared the winner. If the remainder of the board cannot be covered with paperclips then Penny is the winner.

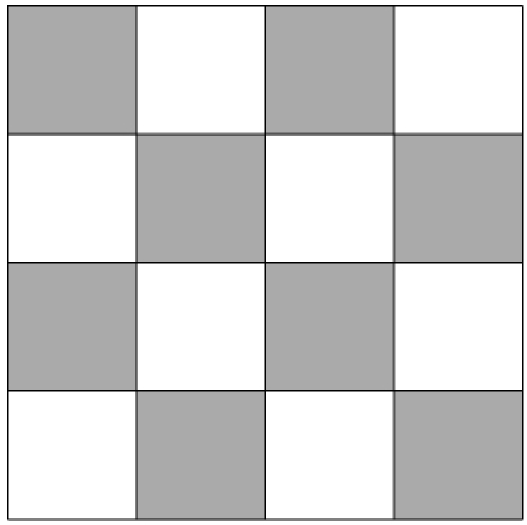


FIGURE 3. Beginner board for Pennies and Paperclips game.

42. With a partner, play this game several different times. Record the results of your games, including the placement of the pennies a paperclips, on the miniature boards in Figure 4.
43. Now switch the roles of pennies and paperclips and play several more games, again recording your results.
44. Do you notice any patterns that will enable you to find winning strategies for the players? If so, test them by playing a few more games. If not, continue to play until some pattern appears.
45. State a conjecture which determines precisely when pennies win based on their placement.
46. Prove this conjecture.
47. State a conjecture which determines precisely when paperclips win based on the placement of the pennies.
48. Prove this conjecture.
49. Are there any situations in which neither player wins, or have you characterized all possible outcomes? Explain.

It is important to remember that a proof is a logically complete demonstration that a result must hold. Proof by example is not allowed. The proof most people originally give for Investigation **48** is merely a counting argument.

50. Play pennies and paperclips a few times on the distorted board in Figure 5.
51. Does your conjecture in Investigation **47** hold for this board?
52. Return to your proof in Investigation **48**. Does your proof fundamentally use the geometry of the board or is it simply a counting argument? The distorted board generally shows that the "proofs" in Investigation **48** are not complete.
53. If your proof in Investigation **48** is not complete, see if you can use the *Hamiltonian circuit* to show that paperclips can be appopriately placed to win the expected games.

The board you have been playing on was called the "beginner" board. Mathematicians love to generalize. It is a very natural mathematical question to ask whether this game can be extended to other sized boards. Try other sized boards and see what you find!

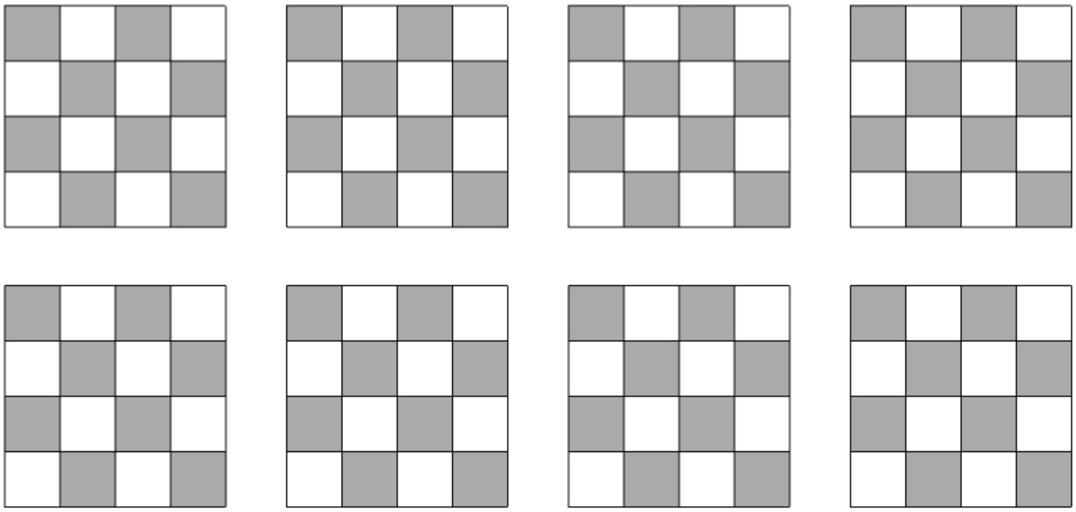


FIGURE 4. Board for recording Pennies and Paperclips games.

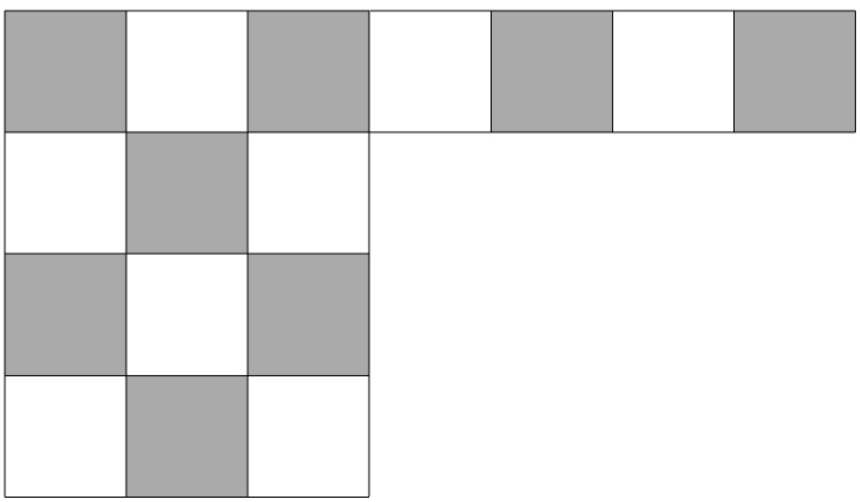


FIGURE 5. Distorted beginner board for Pennies and Paperclips game.

4.4. Monty Hall Problem.

> A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks.
>
> **Gian-Carlo Rota** (; -)

This is good because it is so contrary to our intuition. This is one of the reasons that we really need proof.

Good back story too.

How to do it? Data collection?

CHAPTER 8

Teacher Materials

It is important to respect the fundamental way we are asking the traditional classroom experience in mathematics to be replaced with a new pedagogy. If you are reading this as a teacher you have already made a commitment to trying to make this change. But this change will not be natural at all to the students who are mostly used to traditional classrooms.

In our experience the students who have struggled in their previous mathematics courses will be more at ease with this transition. Those who have previously been more successful will be much more resistant. They have learned how to operate in a system where they mimic and regurgitate what they have been shown. You are asking them a great deal to give up the relative comfort they have found.

We find it helpful to deal with this challenge explicitly after the students have experienced several days of inquiry.

One approach is to have each student reflect on the following prompt: I learn best when...

(Need some blah PKH.)

1. Proof

The proof of the chisenbop method by mathematical induction is quite compelling to the students.

Theorem (Remainder by Nines) The remainder when a positive integer is divided by 9 is the sum of its digits.

Generally one has to sum the digits repeatedly, something students will quickly notice. Summing the digits of 5283217 is 28, which is a remainder but not an appropriate one since 9 can still be divided into this remainder. Three times with remainder 1, which is the same as summing the digits 10 and then summing them again, 1.

Typically students are hesitant to get actively working on this problem because they expect some solution to magically appear. They often need data collection to be suggested. Once collecting, some students will certainly do this systematically finding the remainders of $1, 2, 3, \ldots$ follow the pattern $1, 2, 3, 4, 5, 6, 7, 8, 0, 1, 2, 3, \ldots$ They have rediscovered arithmetic modolo 9. This is a great and useful pattern, and deserves respect. But it does not provide an efficient way to determine the remainder - ask them, "What does this tell you about the remainder when 5937120 is divided by 9?

Several different proofs are illustrated in the explicit case of 548 below:

Base Ten Blocks

In Figure 1 one sees the standard pattern blocks with the maximum number of 9s removed from each block, clearly leaving remainders of $5 + 4 + 8$. Easy to see how this can be generalized.

Expanded Notation and Commutative Law Perhaps inspired by the Human Calculator trick the expanded notation can provide an arithmetic analogue of the Base Ten Block proof:

$$548 = 5\,(1 + 99) + 4\,(1 + 9) + 8 = 5 + 5 \times 99 + 4 + 4 \times 9 + 8.$$

Clearly 5×99 and 4×9 are divisible by 9, so the remainder of 548 upon division by 9 is the remainder of $5 + 4 + 8$. Since this remainder is greater than 9, one divides 9 into it once more to find the resultant remainder of 8.

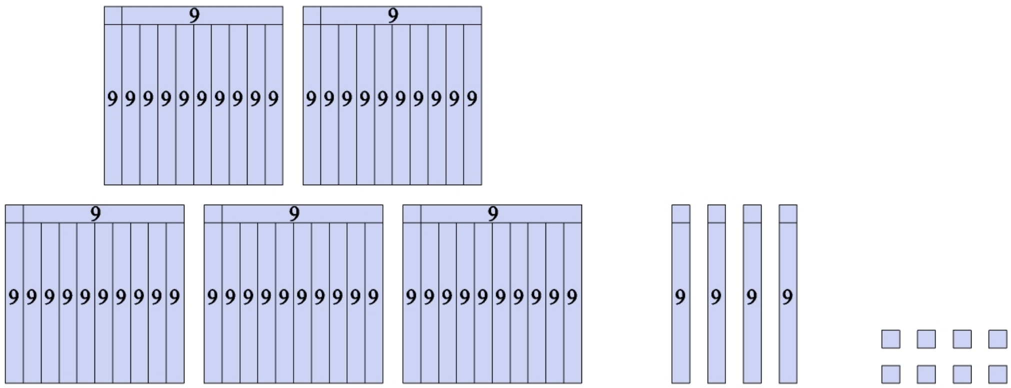


FIGURE 1. Remainder when 548 is divided by 9, base ten block style.

Expanded Notation and Remainder for Powers of Ten In their data collection students often realize that the remainder of each power of 10 is one. Alternatively, they see that $10, 20, 30, \ldots 90$ have remainders $1, 2, 3, \ldots 0$ and similarly $100, 200, 300, \ldots 900$ have remainders $1, 2, 3, \ldots, 0$. Since

$$548 = 5 \times 100 + 4 \times 10 + 8 = 500 + 40 + 8$$

the remainder is certainly $5 + 4 + 8 = 17$ which they recognize as the remainder of 8. Generalizing is immediate.

For a very nice description of activity-based investigations which extend the multiple of 9s finger counting considered above, see "Let Your Fingers Do the Multiplying," Sidney J. Kolpas, *Mathematics Teacher*, Vol. 95, No. 4, April, 2002, pp. 246-51.

Casting out Nines is nicely described in "Casting Out Nines Revisited" by Peter Hilton and Jean Pedersen in *Mathematics Magazine*, Vol. 54, No. 4 (Sep., 1981), pp. 195-201 and "Casting Out Nines: Our Decimal Number System" by Louis E. Ulrich *The Arithmetic Teacher*, Vol. 2, No. 3 (OCTOBER 1955), pp. 77-79.

1.1. $3a+5b$ **Problem.** The different patterns that students found on which to base their proofs are quite remarkable if left to their own devices. Four are shown below.

1.2. Pythagorean Theorem. In doing Bhaskara's proof of the Pythagorean theorem, students will often use the triangles to frame an empty square with sides c. In other words, the triangles have been reflected outside of the square rather than inside as Bhaskara had them, as shown in Figure 6. This proof is as old as the 200 BCE from Chou Pei Suan Ching, one of the oldest and most well-known surviving Chinese mathematical texts. It is a perfectly fine alternative proof.

In fact, there are other proofs that can be devised from this set-up and students should be encouraged to find alternatives and not sticking directly to the scripted investigations.

1.3. Pennies and Paperclips. It is essential that students understand that to say "When pennies are on opposite colors there are 7 white squares and 7 black squares so Clips can place their 7 paperclips to cover these squares" is *not* a proof. This argument *does not* work on the distorted gameboard, illustrating a limitation of the "proof."

There are many ways to extend this game and to devise problems/exercises that can be used for additional work or assessment. Some details about these extensions are as follow:

- On a 5×5 board (or, in fact, any board that is odd by odd) pennies will always win as there will be an uneven number of squares for paperclips to cover. However, one can make this a non-trivial game by requiring pennies to place 3 (or any other odd number) of pennies.

10/10

Nice!

Flows better that way?

3a+5b conjecture: Beginning with 8, any whole number may be a solution to 3a+5b (given that a,b > 0 or =0), in addition to the numbers 0, 3, 5, and 6. The numbers 1, 2, 4, and 7 cannot be solutions to 3a+5b when a and b are greater than or equal to zero.

0, 3, 5, and 6 can be found be setting both a and b equal to zero, setting a equal to 1 and b equal to zero, setting a equal to zero and b equal to 1, and setting a equal to 2 and b equal to zero, respectively. 1, 2, and 4 cannot be solutions because they are less than either 3 or 5, while 7 cannot be a solution because 3 and 5 are both less than 7, but combined are greater than 7.

When 3a+5b=8, a=1 and b=1. From here you can use 3a+5b to increase the solution by 1 using the following algorithm:

- Increase a by 2 (add two 3s), decrease b by 1 (subtract one 5)
- Decrease a by 3 (subtract three 3s), increase b by 2 (add two 5s)
- Increase a by 2, decrease b by 1
- Repeat

Using the above pattern, the solution to 3a+5b will increase by 1 with each successive step completed, allowing for infinite repetition to find every whole number greater than 8 as a solution. By adding two 3s and subtracting one 5, you are adding 6 and subtracting 5, so in the end 1 is the net increase (likewise with subtracting three 3s and adding two 5s, you are subtracting 9 while adding 10, so the increase is still 1). This algorithm of adding and subtracting multiples of 3 and 5 will always work because by starting at 8, a=1 and b=1. Over one cycle of the algorithm, a will increase by a total of 1, while b nets no gain:

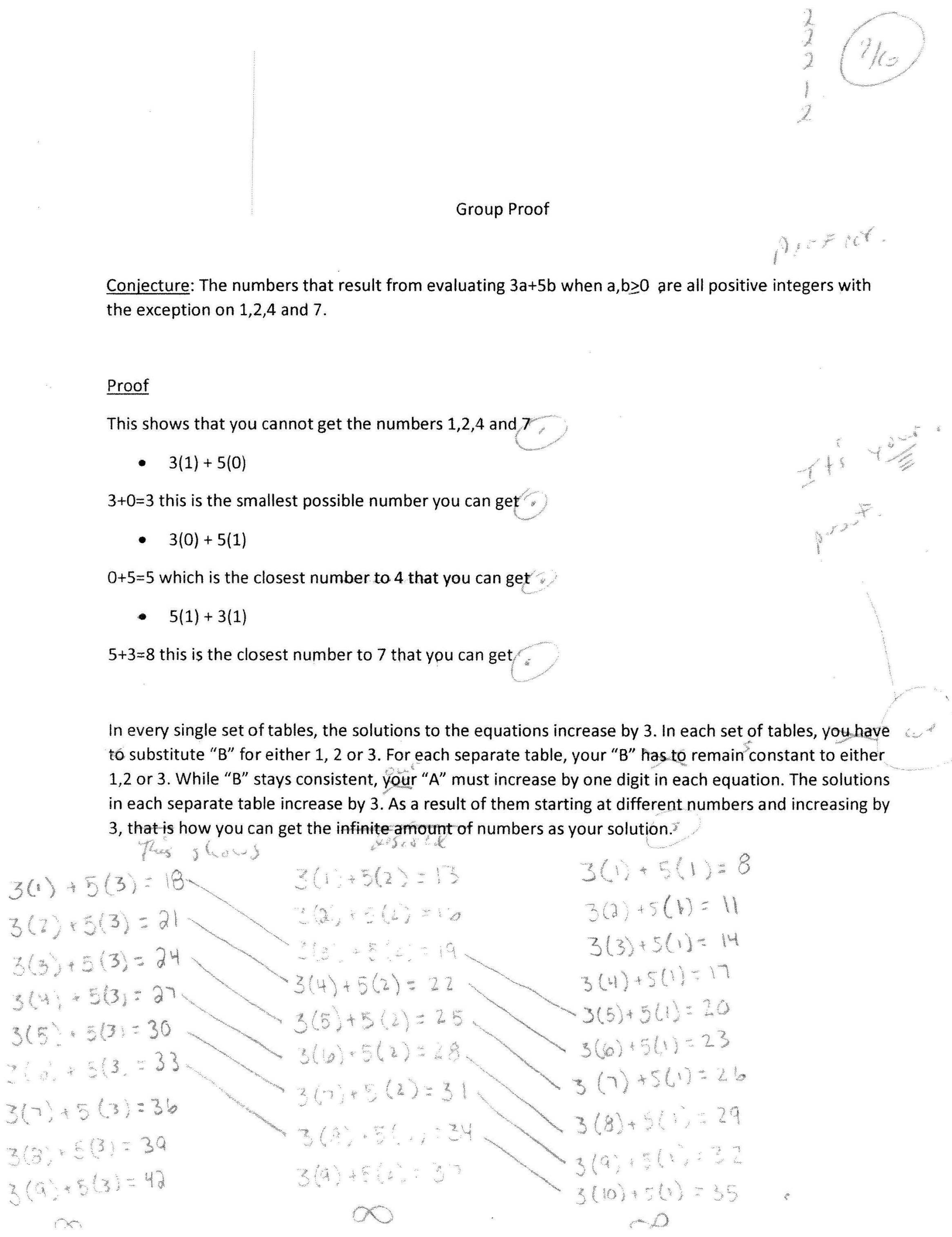
	A	B
8	1	1
	+2	*-1*
9	3	0
	-3	*+2*
10	0	2
	+2	*-1*
11	2	1

Since the lowest a reaches in the first cycle is 0, and it increases by 1 each cycle, it will never dip below 0. Since b nets no increase each time, it will cycle through 1, 0, and 2.

Perfect!

Figure 2. Proof by algorithm of the $3a + 5b$ problem.

- On a 6×6 board (or, in fact, any other board that is even by even) the game is played just as it is on a 4×4 board with the same outcomes and similar proofs.


2
2
2
1
2
9/10

Group Proof

Perfect.

Conjecture: The numbers that result from evaluating 3a+5b when a,b≥0 are all positive integers with the exception on 1,2,4 and 7.

Proof

This shows that you cannot get the numbers 1,2,4 and 7

- 3(1) + 5(0)

3+0=3 this is the smallest possible number you can get

- 3(0) + 5(1)

0+5=5 which is the closest number to 4 that you can get

- 5(1) + 3(1)

5+3=8 this is the closest number to 7 that you can get

It's your proof.

In every single set of tables, the solutions to the equations increase by 3. In each set of tables, ~~you have~~ to substitute "B" for either 1, 2 or 3. For each separate table, your "B" ~~has to~~ remain constant to either 1,2 or 3. While "B" stays consistent, your "A" must increase by one digit in each equation. The solutions in each separate table increase by 3. As a result of them starting at different numbers and increasing by 3, ~~that is~~ how you can get the ~~infinite amount~~ of numbers as your solution.

This shows

3(1)+5(3)=18
3(2)+5(3)=21
3(3)+5(3)=24
3(4)+5(3)=27
3(5)+5(3)=30
3(6)+5(3)=33
3(7)+5(3)=36
3(8)+5(3)=39
3(9)+5(3)=42
∞

3(1)+5(2)=13
3(2)+5(2)=16
3(3)+5(2)=19
3(4)+5(2)=22
3(5)+5(2)=25
3(6)+5(2)=28
3(7)+5(2)=31
3(8)+5(2)=34
3(9)+5(2)=37
∞

3(1)+5(1)=8
3(2)+5(1)=11
3(3)+5(1)=14
3(4)+5(1)=17
3(5)+5(1)=20
3(6)+5(1)=23
3(7)+5(1)=26
3(8)+5(1)=29
3(9)+5(1)=32
3(10)+5(1)=35
∞

FIGURE 3. Proof by mod 3 equivalence classes of the $3a + 5b$ problem.

- One can generalize to boards of any size and shape. The results on the simple boards considered above will hold for larger, irregularly shaped boards as long as the parity (number of black versus white squares) is retained and there is a Hamiltonian circuit covering the

2
2
2
0
2
8/10

Conjecture

Using the expression 3a+5b you can get any positive integer as an answer besides 1,2,4,7 as long as a and b > 0.

to what b?

Proof

For every number you multiply 3 by a pattern, (1,0,2,1,3) for the 5 there is also a pattern. This pattern however changes it starts with (0,1,0,1,0) for the first five numbers. Then the pattern changes to (2,1,0,2,1). It then keeps this general pattern and then add 1 to those five digits for the next 5 numbers. This pattern repeats infinitively. You cannot get the numbers 1,2,4, and 7 because 3 and 5 or 3+5 are not factors of 1,2,4, and 7.

Infinitely

Factors are for multiplying.

Example:

3a+5b

3(1) + 5(0) = 3	
3(0) + 5(1) = 5	
3(2) + 5(0) = 6	
3(1) + 5(1) = 8	
3(3) + 5(0) = 9	
3(0) + 5(2) = 10	Add 10, subtract 9 – subtracting three 3's and then adding two 5's
3(2) + 5(1) = 11	Add 6, subtract 5 – subtracting one 5 and then adding two 3's
3(4) + 5(0) = 12	Add 6, subtract 5 – subtracting one 5 and then adding two 3's
3(1) + 5(2) = 13	Add 10, subtract 9 – subtracting three 3's and then adding two 5's
3(3) + 5(1) = 14	Add 6, subtract 5
3(0) + 5(3) = 15	Add 10, subtract 9
3(2) + 5(2) = 16	Add 6, subtract 5
3(4) + 5(1) = 17	Add 6, subtract 5
3(1) + 5(3) = 18	Add 10, subtract 9
3(3) + 5(2) = 19	Add 6, subtract 5

For this pattern you are adding one to the sum by trading numbers in. For every 5 you take away you are adding 2 3's, for every 3 3's you take away you are adding 2 5's. By trading in numbers using this pattern, you are always adding one. This pattern works because you will always have enough 5s since every 5th number is multiple of 5. Every time you reach a multiple of 5 you will have enough 5s. You can continue this pattern infinitely.

This is the key to your proof!

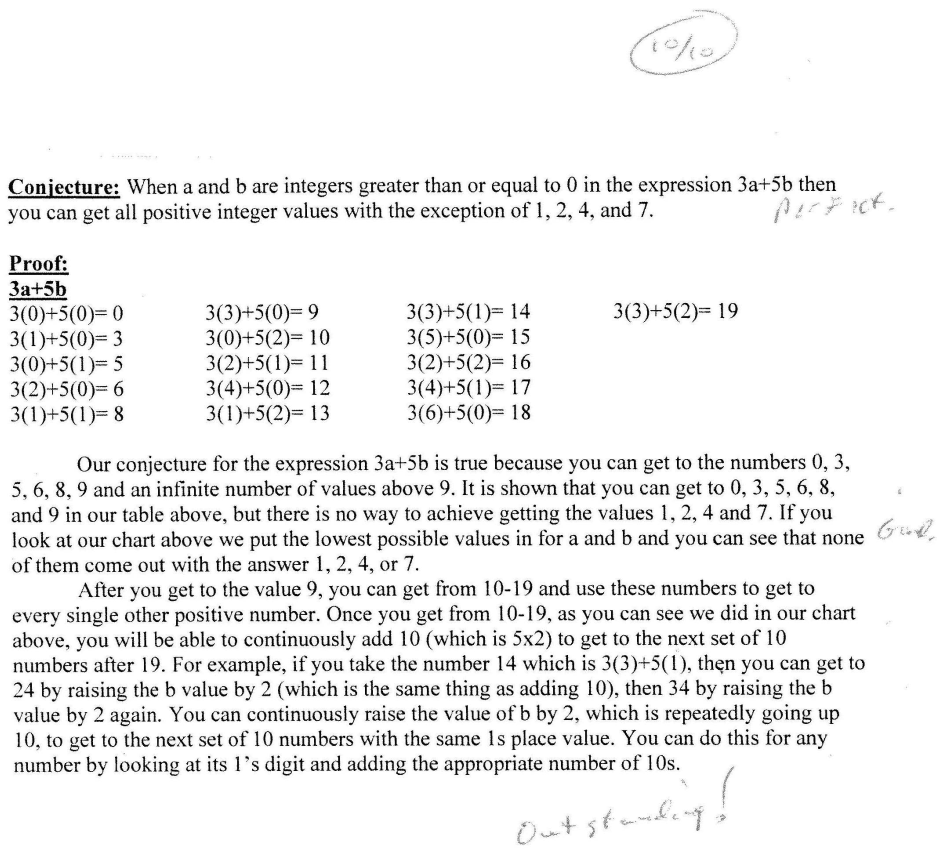
to start this process again.

Figure 4. Proof of the $3a+5b$ problem. Note how the language they use was deeply informed by the inductive proof of the Chisenbop multiple of nine result.

entire board which provides for the placement of the paperclips *and* which respects the parity.[1]

- One can show portions of boards with particular arrangements of pennies and ask whether there are winning strategies for paperclips in an unlimited numbers of ways. See for example the portions of gameboards in Figure 7

[1]See "Tiling with Dominoes" by N.S. Mendelsohn, *College Mathematics Journal*, Vol. 35, No. 2, March 2004, pp. 115-20.



10/10

Conjecture: When a and b are integers greater than or equal to 0 in the expression 3a+5b then you can get all positive integer values with the exception of 1, 2, 4, and 7. Perfect.

Proof:
3a+5b

3(0)+5(0)= 0	3(3)+5(0)= 9	3(3)+5(1)= 14	3(3)+5(2)= 19
3(1)+5(0)= 3	3(0)+5(2)= 10	3(5)+5(0)= 15	
3(0)+5(1)= 5	3(2)+5(1)= 11	3(2)+5(2)= 16	
3(2)+5(0)= 6	3(4)+5(0)= 12	3(4)+5(1)= 17	
3(1)+5(1)= 8	3(1)+5(2)= 13	3(6)+5(0)= 18	

Our conjecture for the expression 3a+5b is true because you can get to the numbers 0, 3, 5, 6, 8, 9 and an infinite number of values above 9. It is shown that you can get to 0, 3, 5, 6, 8, and 9 in our table above, but there is no way to achieve getting the values 1, 2, 4 and 7. If you look at our chart above we put the lowest possible values in for a and b and you can see that none of them come out with the answer 1, 2, 4, or 7. Good.

After you get to the value 9, you can get from 10-19 and use these numbers to get to every single other positive number. Once you get from 10-19, as you can see we did in our chart above, you will be able to continuously add 10 (which is 5x2) to get to the next set of 10 numbers after 19. For example, if you take the number 14 which is 3(3)+5(1), then you can get to 24 by raising the b value by 2 (which is the same thing as adding 10), then 34 by raising the b value by 2 again. You can continuously raise the value of b by 2, which is repeatedly going up 10, to get to the next set of 10 numbers with the same 1s place value. You can do this for any number by looking at its 1's digit and adding the appropriate number of 10s.

Outstanding!

FIGURE 5. Proof by digit analysis of the $3a + 5b$ problem.

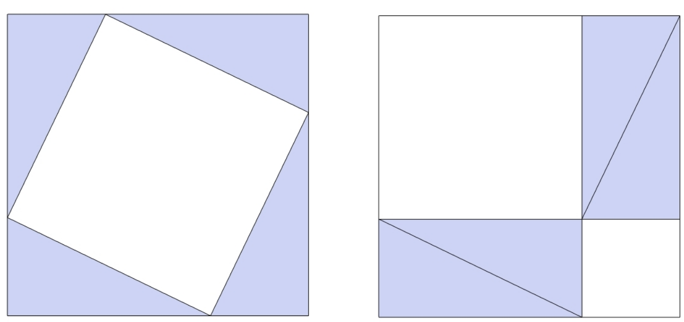


FIGURE 6. Chou Pei Suan Ching proof of the Pythagorean theorem

[Quote about being apprenticeshipped? From Krantz proof book.]

Students are being asked to prove things. Most students expect there to be a method, algorithm, recipe or style sheet to follow - perhaps like the two-column proofs required in many high school geometry courses. But the idea of proof is much broader than that. It is critical to illustrate the diversity of what proof. We have tried to do this in the investigations. But it is also important to

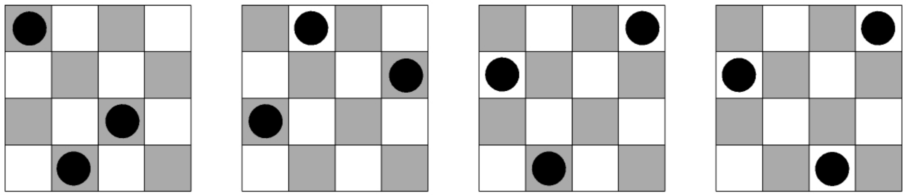


Figure 7. Distorted beginner board for Pennies and Paperclips game.

model different proofs for the students. The may feel that their reasoning doesn't really constitute a proof because its form does not mimic one that they have legitimized by an authority figure. Illustrating a diverse collection of proofs helps.

Proof underlies the majority of the material in each volume in the Discovering the Art of Mathematics series. So examples abound. Some that we find work particularly well because they require very little background are:

- $0.999\ldots = 1$.
- Insert more here.

1.4. Sample Conjecture and Proof. Below is a sample statement of the result on Galileo fractions and its proof.

Notice the important cycle here. In class we have been exploring, investigating, seeking to understand how these fractions work. To borrow the language of theatre and narrative writing, we reach a climax when we state our conjecture. The proof is akin to the denouement or resolution. When the proof is complete, the conjecture takes the status of a theorem - a mathematical truth now eternally established.

Theorem (Galileo fractions) Formed from the consecutive odd numbers beginning at 1, all of the fractions $\frac{1}{3}, \frac{1+3}{5+7}, \frac{1+3+5}{7+9+11}, \frac{1+3+5+7}{9+11+13+15}, \ldots$ are equal to $\frac{1}{3}$.

Proof Adding the numerator and the denominator together gives a sum of consecutive odds beginning at 1. This is a Galileo sum. Using the method from the proof of the Galileo sums theorem, we can represent this sum of numerator and denominator combined as nested ⌟ shapes, one for each odd number in our sum, to form a square as shown in Figure 8 below. There are the same number of terms in the numerator and denominator. This means the dimensions of the square are even and it can be divided in half both vertically and horizontally as shown.

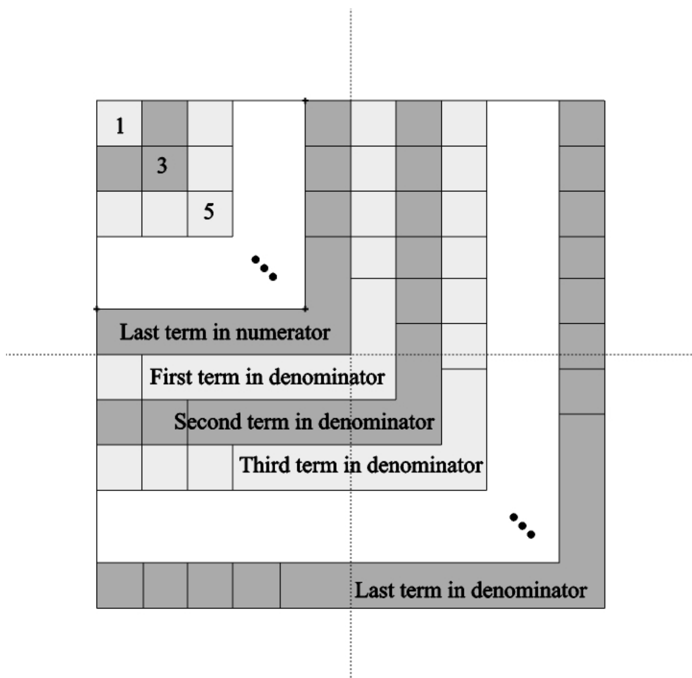


FIGURE 8. Galileo fractions - Proof Without Words.

By construction, the upper left-hand quarter of this square is precisely the sum of the terms in the numerator. Similarly, the remainder of the square is precisely the sum of the terms in the denominator. Since this is three-quarters of the square, it is exactly three times the upper left-hand corner. I.e. the sum of the terms in the denominator is three times larger than the sum of the terms in the numerator. Therefore, the value of the Galileo fraction must be $\frac{1}{3}$.//

For students fluent with algebra, there is a shorter algebraic proof which brings up very interesting questions about the role of proof. Namely, the result of Galileo sums is $1+3+5+\ldots+(2n-1) = n^2$. We can write the general Galileo fraction as follows:

$$\begin{aligned}
\frac{1+3+5+\ldots+(2n-1)}{(2n+1)+(2n+3)+\ldots+4n-1} &= \frac{1+3+5+\ldots+(2n-1)}{(1+3+5+\ldots+(4n-1))-(1+3+5+\ldots+(2n-1))} \\
&= \frac{n^2}{(2n)^2-n^2} \\
&= \frac{n^2}{3n^2} \\
&= \frac{1}{3}.
\end{aligned}$$

For many this will be mystical, algebraic sleight of hand while the proof without words gives a clear, holistic, intuitive resolution. The idea behind this original *proof without words* is from

"Proof Without Words: Galileo's Ratios Revisited" by Alfino Flores and Hugh A. Sanders, *College Mathematics Journal*, vol. 36, no. 3, May 2005, p. 198.