Inquiry-Based Class Tackles Rascals' Triangle

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Discovery consists in looking at the same thing as everybody else and seeing something different. A. Szent-Gyorgyi

The question is not what you look at, but what you see. H.D. Thoreau

The real voyage of discovery consists not in seeking new landscapes but in having new eyes. Marcel Proust

1 Introduction

As part of our semester-long study of patterns, our inquiry-based mathematics for liberal arts class was intrigued by Rascals' triangle.[1] We saw something different and here we offer new eyes with which to see the triangle.

2 Rascals' Triangle

In "The Rascal Triangle" [1] the three authors, at that time one a seventh grader and two eighth graders, tell of challenging a question from an I.Q. test which asked for the next row in the triangular array that is known as the first four rows of Pascal's triangle, as shown in Figure 1.

Instead of the well-known Pascal's triangle, the authors invented Rascals' Triangle shown in Figure 2.

As the defining rule of Pascal's triangle can be described by an "inverted triangle formula" $West + East = South$ the Rascals discovered that their triangle seemed to have a "diamond shape" defining rule

$$\frac{West \times East + 1}{North} = South.$$

[1] Our class is Mathematical Explorations at Westfield State University. We are Alexis Bates, Mitchell Begin, Brianna Cora, Darryl Denson, Mary Donnelly, Cameron Earle, Elizabeth Gallagher, Edward Gard, Brandon Hammer, Caitlyn Hurley, Ross Lewicki, Devorah Lipschitz, Thomas Moore, Constance Morgan-Poirier, Barbara Nguyen, Matthew O'Connor, Katherine Parent, Taylor Powers, Amanda Sheldon, Courtney Smith, Jonathan Tumblin, Cameron Vujs, Miranda Weathers, Jonathan Whalen and Julian F. Fleron. Professor Fleron's work on Discovering the Art of Mathematics (www.artofmathematics.org), which provides the curriculum for this course, is supported by NSF 1225915.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ ? & & ? & & ? & & ? & & ? \end{array}$$

Figure 1: I.Q. test question - finding the fourth row of Pascal's triangle.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 5 & & 4 & & 1 & & \\ & 1 & & 5 & & 7 & & 7 & & 5 & & 1 & \\ 1 & & 6 & & 9 & & 10 & & 9 & & 6 & & 1. \end{array}$$

Figure 2: The first seven rows of Rascals' triangle.

The Rascals were challenged by their teaching to prove their rule was correct since a formula involving division could result in a non-integer. In their paper the Rascals describe how they proved the correctness of their formula by noticing that the n^{th} entry in column m of their triangle is given by:

$$mn + 1. \tag{1}$$

With some non-trivial algebra they then showed

$$\begin{aligned} \frac{West \times East + 1}{North} &= \frac{(m(n+1)+1)((m+1)n+1)+1}{mn+1} \qquad (2) \\ &= (m+1)(n+1)+1 \qquad (3) \\ &= South. \qquad (4) \end{aligned}$$

3 New Eyes Looking Upon Rascals' Triangle

Our course focused on patterns. We were encouraged to find patterns whenever we could, to analyze these patterns, to uncover where these patterns come from and prove that they continued. We referred to this process of pattern formation "morphogenesis."

As we investigated Rascals' triangle one of our groups found a rule. This is not what the professor expected. He didn't tell us we were wrong, but walked away shaking his head, clearly surprised by our discovery. Soon after another group discovered the same rule. And then another. But then another group, which had been stumped, looked up the Rascals' rule online. And it was different!

We tested our new rule on many different cases and it seemed correct. It appeared there were two totally different rules. How can there be two totally different answers? Which one is correct?

Our new rule was:

$$West + East + 1 = North + South.$$

As we continued testing our formula we became convinced our rule was correct. And then we found a proof:

$$\begin{aligned}
West + East + 1 &= (m(n+1)+1) + ((m+1)n+1) + 1 && (5)\\
&= mn + m + 1 + mn + n + 1 + 1 && (6)\\
&= (mn+1) + (mn + m + n + 1 + 1) && (7)\\
&= (mn+1) + ((m+1)(n+1)+1) && (8)\\
&= North + South. && (9)
\end{aligned}$$

So both rules are correct!

While this proved the formula, we wanted to understand the "morphogenesis" of this pattern - where it came from and why it arose.

The essential property of Rascal's triangle is that the data in each column is linear and the first differences in each column increase by 1 in each successive column. In fact, that is where the defining equation in (1) comes from. And then a great idea hit us! The two sides in the "diamond shape" are consecutive, corresponding terms along successive columns. This means the first differences along one side are simply one more/less than they are on the other. In other words,

$$West - North + 1 = South - East.$$

Rearranging we have our formula: $West + East + 1 = North + South$. *This* is where this formula really comes from.

We have not found any evidence that this easier rule has been noticed before.[2]

Referring back to the proof that the general term in Pascal's triangle is $\binom{r}{m} = \frac{r!}{(r-m)!m!}$ the Rascals' ended their paper by asking "Which triangle is simpler now?"

In the same spirit of having new eyes, we think the Rascals would like our question: Who's formula is simpler now?

4 Conclusion

Rascals' triangle and the new, simpler formula for its entries that is presented here speak to the abilities of students to make substantive discoveries when provided the opportunity to freely explore. Their "new eyes" often see things that their teachers have been conditioned not to see because they "know" the answers already.

[2]Our teacher found cool Frieze patterns made by numbers generated by rules that use both a diamond addition pattern and a diamond multiplication pattern in [2]. But there is not a direct connection to Rascals' triangle nor any indication that the patterns so generated can be the same.

References

[1] Alif Annggoro, Eddy Liu and Angus Tulloch. "The Rascal Triangle" *College Math. J.*, **41** (2010) 393-395.

[2] John H. Conway and Richard K. Guy. *The Book of Numbers*. Springer-Verlag, New York, 1996. 74-76.