

The following is excerpted from:

## Discovering the Art of Mathematics: Games & Puzzles

By Volker Ecke and Christine von Renesse with Julian F. Fléron and Philip K. Hotchkiss

As with all of our learning guides, this book is freely available online at

<http://www.artofmathematics.org/books/>

*Discovering the Art of Mathematics* (DAoM) is an NSF supported project that supports inquiry-based learning (IBL) approaches for mathematics for liberal arts (MLA) courses.

The DAoM curriculum consists of a library of 11 inquiry-based learning guides. Each volume is built around deep mathematical topics and provides materials which can be used as content for a semester-long, themed course. These materials replace the typical lecture dynamic by being built on inquiry-based investigations, tasks, experiments, constructions, data collection and discussions.

DAoM also provides a wealth of resources for mathematics faculty to help transform their courses. Extensive online resources include volume specific teacher notes and sample solutions, classroom videos of IBL in action, sample student work, regular blogs about teaching using IBL and a regular newsletter. Opportunities for supported reviewing and beta testing are also available.

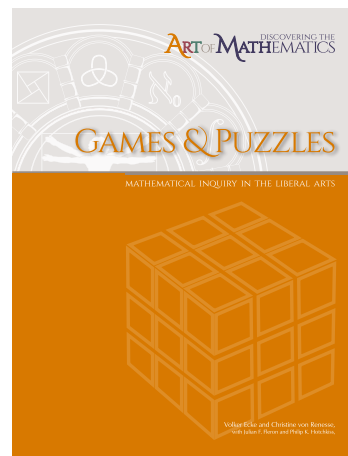
For departments interested in IBL, DAoM offers traveling professional development workshops.

Full information about the *Discovering the Art of Mathematics* project is available at

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Use the excerpt below to explore for yourself how our materials can engage students in mathematical inquiry.

### Radon/Kaczmarz Puzzles

The next puzzles are our own invention and we have chosen to name them **Radon/Kaczmarz puzzles**, or **RK puzzles** for short, in honor of the mathematicians **Johann Radon** (Austrian mathematician; 1887 - 1956) and **Stefan Kaczmarz** (Polish mathematician; 1895 - 1940) whose work we will describe later.\*

Like Sudoku and Kakuro puzzles, RK puzzles involve filling in grids with numbers constrained by certain rules and satisfying certain clues. In RK puzzles the numbers entered into the grid are required to be whole numbers 1 - 9, only now numbers *can* be repeated. The clues for RK puzzles are sums of terms, like in Kakuro, only here the sums take many different forms. We will refer to these sums as **aggregates**. Aggregate data will be given visually, as in Figure 1 below. In this figure the aggregate data and puzzle grid are given on the left, the solved puzzle on the right.

1. Can you solve the RK puzzle #1 in Figure 2 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
2. Can you solve the RK puzzle #2 in Figure 3 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
3. Can you solve the RK puzzle #3 in Figure 4 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
4. Can you solve the RK puzzle #4 in Figure 5 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

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\*After their development, we learned of the *Challenger puzzle* which is quite similar to a 4 by 4 R/K puzzle and which appears in a number of newspaper puzzle columns.

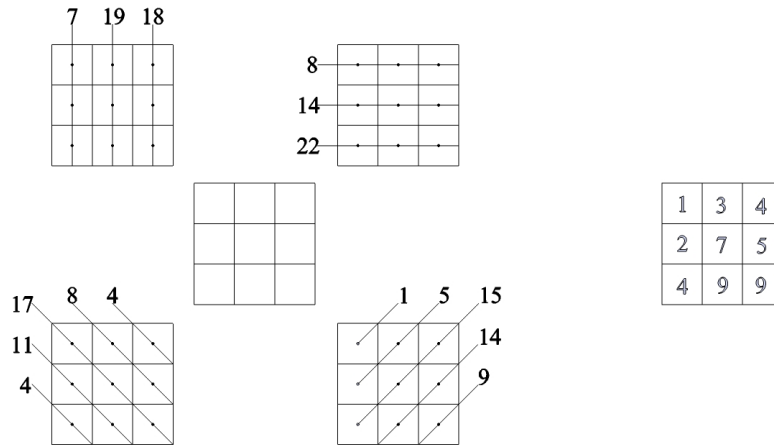


Figure 1: RK puzzle board and aggregate data (left) and solved puzzle (right).

5. Can you solve the RK puzzle #5 in Figure 6 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
6. Can you solve the RK puzzle #6 in Figure 7 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
7. Find simple conditions that are necessary for an RK puzzle to have a solution. These are conditions that can be checked without having to try to explicitly solve the puzzle. These conditions should be robust enough that they explain why any of the insolvable RK puzzles above are in fact not solvable. Prove that these conditions are necessary.
8. Are the conditions in Investigation 7 sufficient conditions for an RK puzzle to have a solution? I.e. if the conditions are satisfied does this guarantee that there is a solution to the RK puzzle? Prove that your result is correct.
9. Make and then prove a *positive result* about the solution of *arbitrary* 3 by 3 RK puzzles where the aggregate data includes the 3 vertical aggregates, the 3 horizontal aggregates, and both the 5 left and 5 right diagonal aggregates.

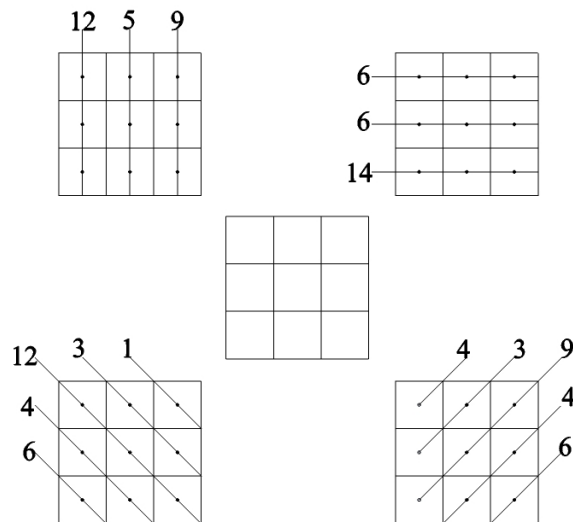


Figure 2: RK puzzle #1.

We are now going to move on to larger RK puzzles.

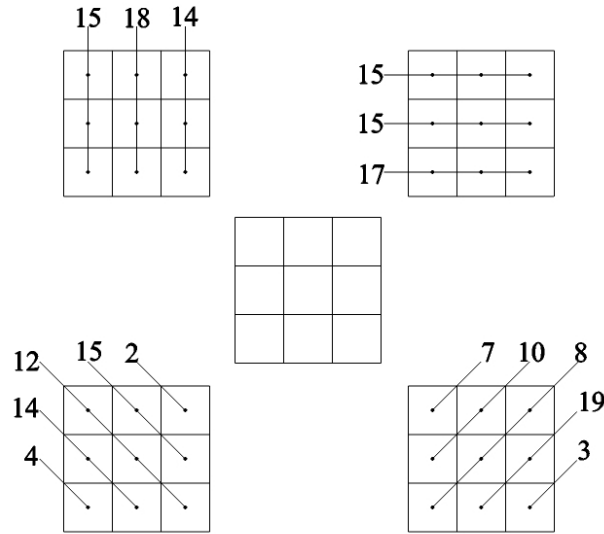


Figure 3: RK puzzle #2.

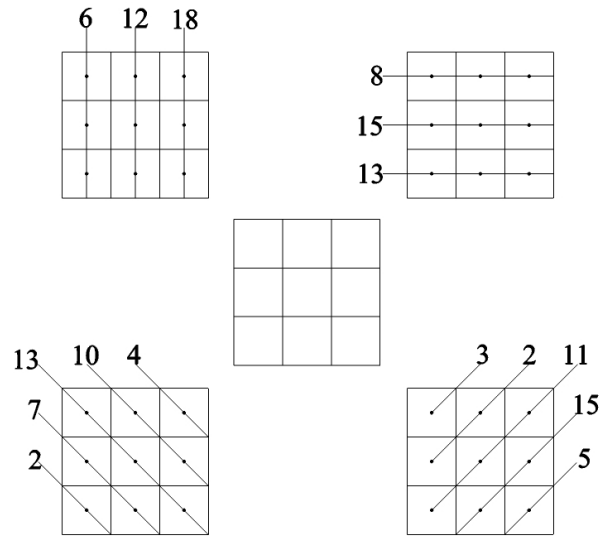


Figure 4: RK puzzle #3.

10. Can you adapt the conditions you found in Investigation 7 to 4 by 4 puzzles? If so, explain how. If not, explain why not. What about RK puzzles that are larger than 4 by 4? Explain.
11. Can you solve the RK puzzle #7 in Figure 8 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
12. Can you solve the RK puzzle #8 in Figure 9 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
13. Can you solve the RK puzzle #9 in Figure 10 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
14. Can you solve the RK puzzle #10 in Figure 11 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.
15. Can you solve the RK puzzle #11 in Figure 12 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

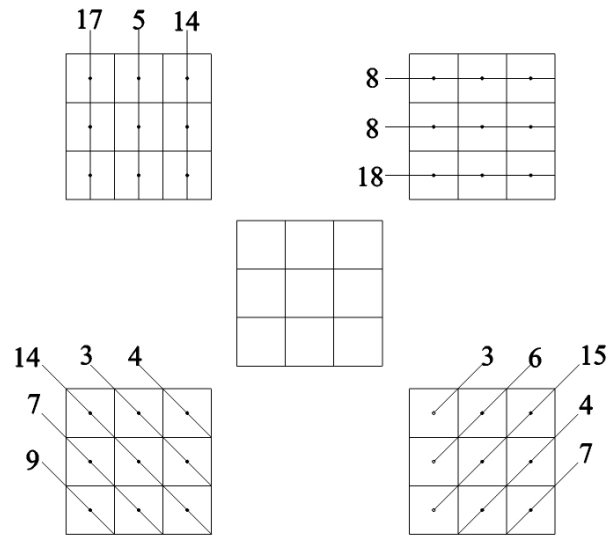


Figure 5: RK puzzle #4.

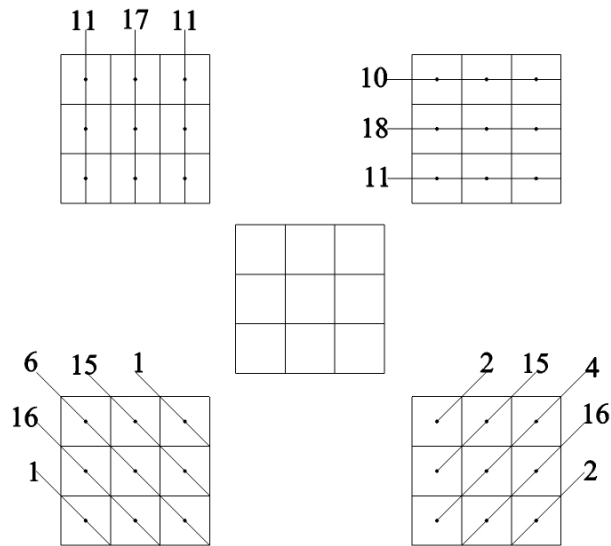


Figure 6: RK puzzle #5.

16. Suppose that you are given the additional *shallow left diagonal* aggregate data, shown in Figure 13, for RK puzzle #11 in Investigation 15. Can you solve the puzzle? If so, is the solution unique? If so, do you need all of the additional data? Whatever your answer, prove that your result is correct.
17. Make and then prove a positive result about the solution of *arbitrary* 4 by 4 RK puzzles where the aggregate data includes the 4 vertical aggregates, the 4 horizontal aggregates, both the 7 left and 7 right diagonal aggregates, and as few shallow left diagonal aggregates as possible.
18. Make your own 4 by 4 RK puzzle whose given aggregate data is: 4 vertical, 4 horizontal, 7 left diagonal, 7 right diagonal, and 10 shallow left diagonals. Explain carefully how you made the puzzle. (Note: Templates are included in the appendix.)

Sudoku, Kakuro, and RK puzzles all assume that there is a unique solution to be had - and one expects sufficient clues to be given or it is not a valid puzzle. In each, our job is to *reconstruct* the solution from incomplete data. Such reconstruction makes puzzles like this part of the important class of problems known as *inverse problems*.

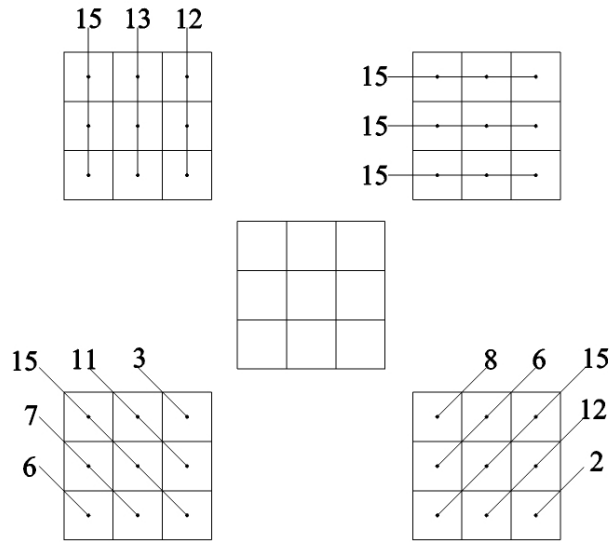


Figure 7: RK puzzle #6.

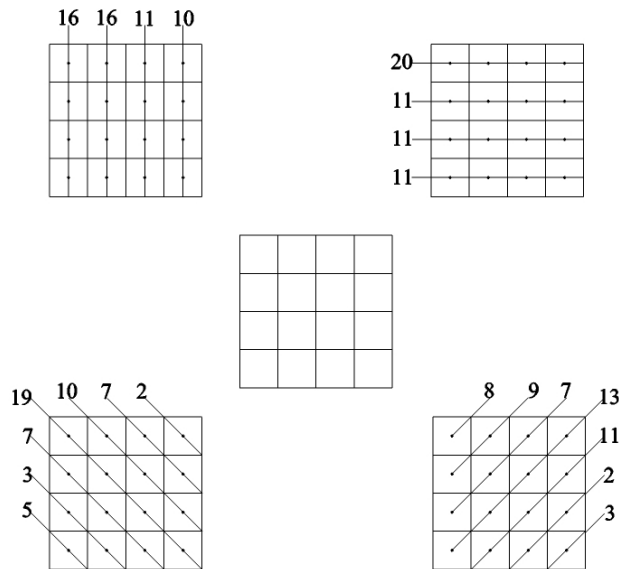


Figure 8: RK puzzle #7.

19. Make your own 5 by 5 RK puzzle whose given aggregate data is: 5 vertical, 5 horizontal, 9 left diagonal, 9 right diagonal, and 13 shallow left diagonals. Explain carefully how you made the puzzle. (Note: Templates are included in the appendix.)
20. Switch 5 by 5 RK puzzles with a partner - giving them only the aggregate data. Were you able to solve the puzzle they gave you with the given aggregate data? Were they?
21. Make and then prove a positive result about *arbitrary* 5 by 5 RK puzzles where the aggregate data includes the 5 vertical aggregates, the 5 horizontal aggregates, both the 9 left and 9 right diagonal aggregates, and the 13 shallow left diagonal aggregates.

In general, having 6 vertical, 6 horizontal, 11 left diagonal, 11 right diagonal, and 16 shallow left diagonal aggregates is not sufficient information to solve a 6 by 6 RK puzzle.

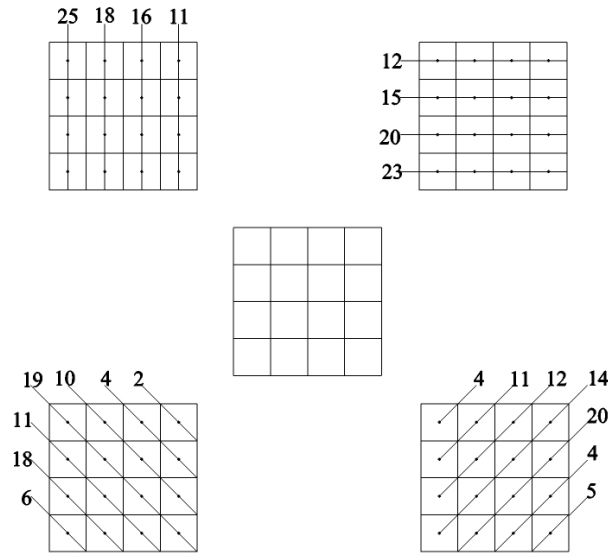


Figure 9: RK puzzle #8.

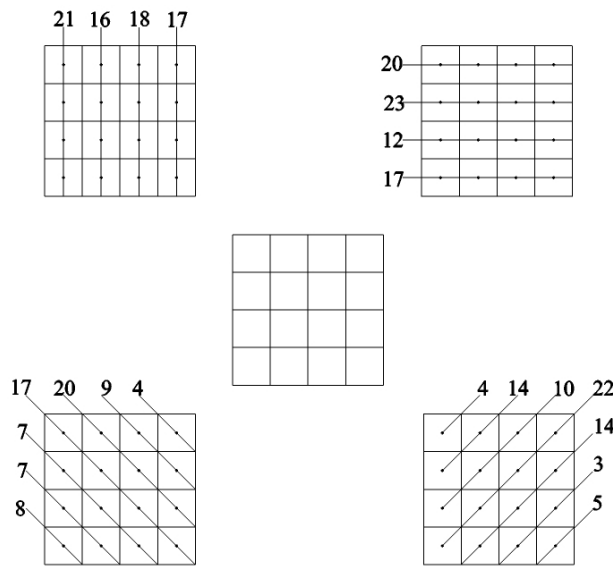


Figure 10: RK puzzle #9.

- 22. Given that such a 6 by 6 RK puzzle is not uniquely solvable, what other aggregates can you think of that might be useful as clues?
- 23. Can you solve the RK puzzle #12 in Figure 14 with the given aggregates? If so, solve it. If not, explain why it cannot be done.

The mathematics behind these puzzles has a fundamental connection to CAT scans and other forms of 3-dimensional medical imaging. To learn more about this connection and its incredible history (about which one leader in the field said, "if Radon [and Kaczmarz] and the early tomographers had been aware of their common problems, many of the [Nobel prize-winning] developments of the past few years might have been launched a half a century ago"), see Chapter 3 - "Other Puzzles: Kakuro, Radon/Kaczmarz, and What s Inside of You."

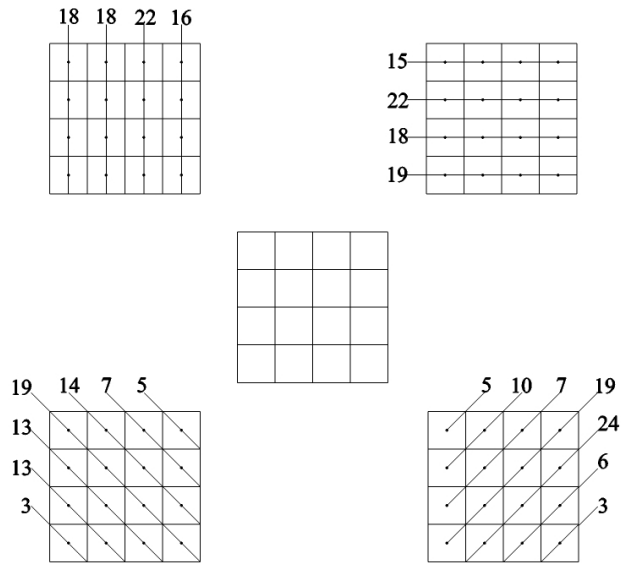


Figure 11: RK puzzle #10.

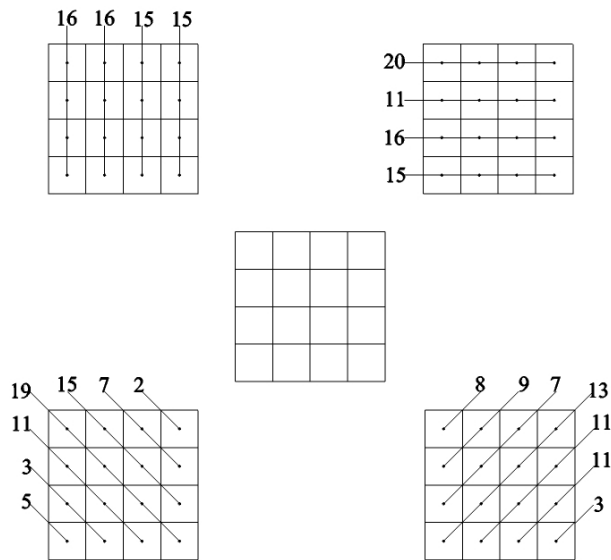


Figure 12: RK puzzle #11.

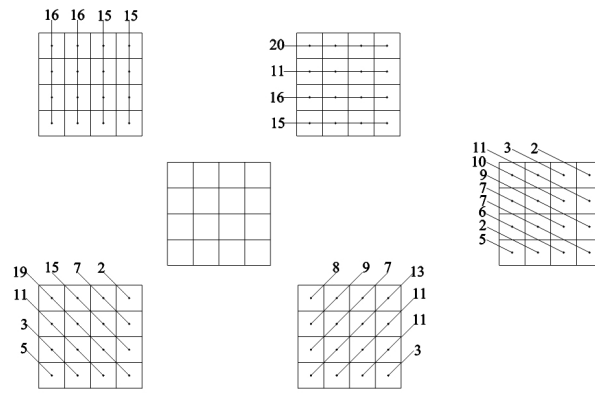


Figure 13: Shallow Left aggregates for RK puzzle #11.

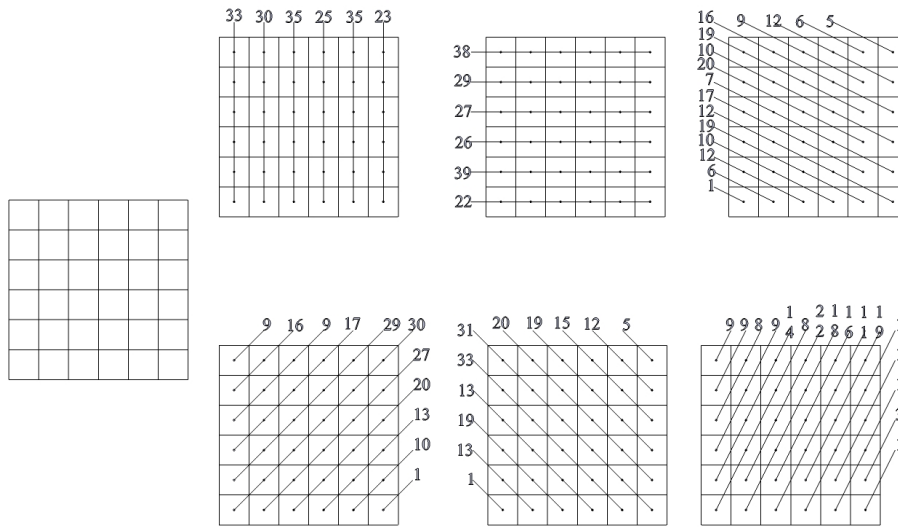


Figure 14: RK puzzle #12.