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MATH 0201 001
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Problem 12: Find a process that allows you to know for sure where the maximum of $P(q_1, q_2)$ is.

$P(q_1, q_2)=(700q_1-q_1^2)+(500q_2-q_2^2)-(16+q_1q_2)$

Domain: $\mathbf{Z}^+$

$0 \le q_1 \le 700$

$0 \le q_2 \le 500$

Solving this problem starts with first finding out what exactly $P(q_1, q_2)$ (or $P(x, y)$ looks like when displayed graphically. We can begin this process by setting $P(x, y)$, x, and y to equal the three axes in a three dimensional space. First I created a table for possible values of x and y.

x	y	P(x, y)
0	0	-16
0	500	-16
700	0	-16
700	500	-350016
500	150	77484
600	0	59984
550	100	67484
525	50	88109
350	250	97484

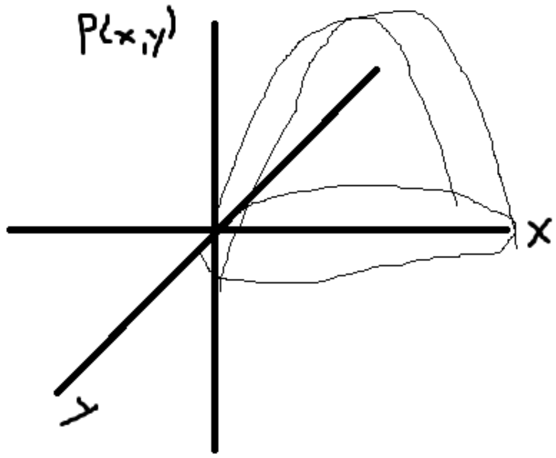
Recording the data for every single possible situation can be a tedious task. The largest number found through this method was P(x, y)=97484; however, this has not yet been proven to be the largest possible P(x, y) value we can find. So is there a better way to find the answer?

Now is when we begin finding out how to graph P(x, y) properly. First we take either x or y and set them to equal a constant. I tried this method several times by setting x to equal 0. Now P(x, y) could be simplified to $P(0, y)=(500y-y^2)-16$. Once again, I recorded several values for the variable y:

y	P(0, y)
0	-16
10	4884
50	22500
100	39984
250	62484
251	62483
500	-16

The shape of this curve can be described as a parabola. It has exactly one maximum or minimum because the form of its equation is quadratic.

Next I tried the same approach, but with y=0 instead. However, since $P(x, 0)=(700x-x^2)-16$ has essentially the same format as P(0, y), I can already tell that it is also a parabola. P(0, y) and P(x, 0) are two perspectives of the same equation, so by laying them on top of one another with respect to their axes in a 3-dimensional space, I will end up with the shape below:



This dome shape is hollow on the inside and only takes into account all positive real values of P(x, y), since negative profits are not considered a part of the range for this problem.

The next question that follows is “how do we find the maximum of the graphed object?” We know what P(x, y) looks like, but we still do not know how to accurately find its peak in any other way than through tedious guessing and checking.

Fortunately there is a way to narrow down our search. Recalling what I learned from Calculus I, I found the derivative of several of the parabola that make up the dome. The derivative of P(x, y)=(500-2y)-c, where x=c, and P(x, y)=(700-2x)-c, where y=c. By setting P’(x, y) to equal 0, we can find exactly what the free variable will be equal to when a specific parabola is at its maximum. First I tried this with x=c:

x	y	P(x, y)
100	200	99984
300	100	139884
400	50	122484

As you can see, we are quickly finding individual maximums that are even greater than the 97484 from earlier.

Now for y=c:

y	x	P(x, y)
150	275	128109
250	225	113109
100	300	129984

Using this method may be quicker, but it still only gives us one parabola at a time, which is still very time-consuming.

To find the maximum of the entire dome faster, we try overlapping the parabolas' derivatives. That is, we can use algebraic substitution between P(x, y)=(500-2y)-c and P(x, y)=(700-2x)-c. These two equations are called the partial derivatives and can help us find the maximum of the whole much faster. The first derivative treats y as the free variable; we call this dp/dy. For the second, x is the free variable; we call this dp/dx. We use substitution between the two equations dp/dy and dp/dx. Because dp/dy dp/dx both equal 0, they are technically the same equation. Next we perform the substitution:

500-2y-x=0

700-2x-y=0

Y=700-2x

500-2(700-2x)-x

=500-1400+4x-x

=-900+3x

X=300

So now we know that the first number x will equal 300 when the dome is at its highest point.

Now by substituting 300 into the original equations we can find y:

$500-2y-300=200-2y=0$

$Y=100$

The highest point on the dome is located at P(300, 100)

Now looking back at when we were experimenting with separate parabolas for y=c, we notice that for y=100, x will equal 300, and P(x, y) will equal 129984, which is the largest value we have found so far.