Fractals III: The Sierpinski Triangle

The Sierpinski Triangle is a figure with many interesting properties which must be made in a step-by-step process; that process is outlined below.

Stage 0: Begin with an equilateral triangle with area 1, call this stage 0, or S_0. (This is pictured below.)

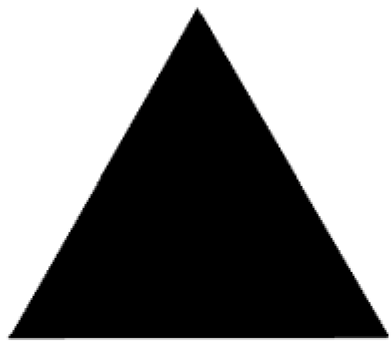

Figure 34: S_0 in the construction of the Sierpinski Triangle.

Stage 1: Now, take S_0 and divide it into equal sized pieces, and then remove one of them, call this result S_1. (The result after the piece is removed is pictured below.)

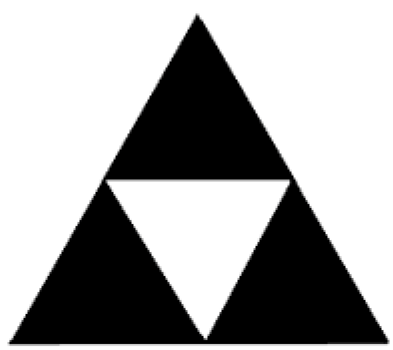

Figure 35: S_1 in the construction of the Sierpinski Triangle.

Stage 2: Now, take each of the equal sized pieces from S_1 (one of which you removed to get S_1) and divide each of those pieces into equal sized pieces, and remove one of them from each of the pieces you had in S_1, call the result S_2. (The result after the pieces are removed is pictured below.)

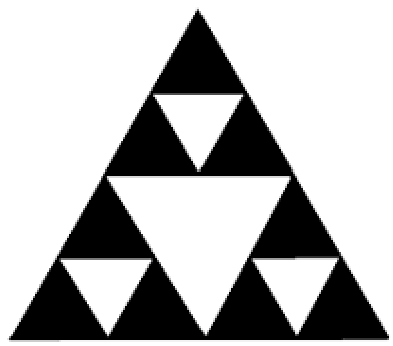

Figure 36: S_2 in the construction of the Sierpinski Triangle.

Stage 3: Now, take each of the equal sized pieces from S_2 (some of which you removed in order to get S_2) and divide each of those pieces into equal sized pieces, and remove one of them from each of the pieces you had in S_2, call the result S_3. (The result after the pieces are removed is pictured below.)

Figure 37: S_3 in the construction of the Sierpinski Triangle.

The Sierpinski Triangle is constructed by repeating this process through an infinite number of stages (see `http://en.wikipedia.org/wiki/File:Animated_construction_of_Sierpinski_Triangle.gif` for a graphic animating the construction through many more stages than we have explored above). It turns out (for reasons we cannot get into) that there is indeed a shape that is made at the end of this (infinite) process. Below is a diagram with the stages we have discussed above (and one more), placed side-by-side

Figure 38: The initial 5 stages, S_0, S_1, S_2, S_3, S_4 in the construction of the Sierpinski Triangle.

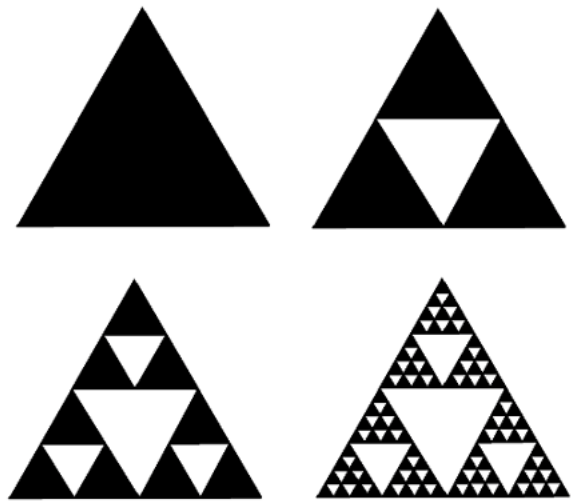

Figure 39: The initial 5 stages, S_0, S_1, S_2, S_3, S_4 in the construction of the Sierpinski Triangle.

See below a larger scale stage in the construction of the Sierpinski Triangle.

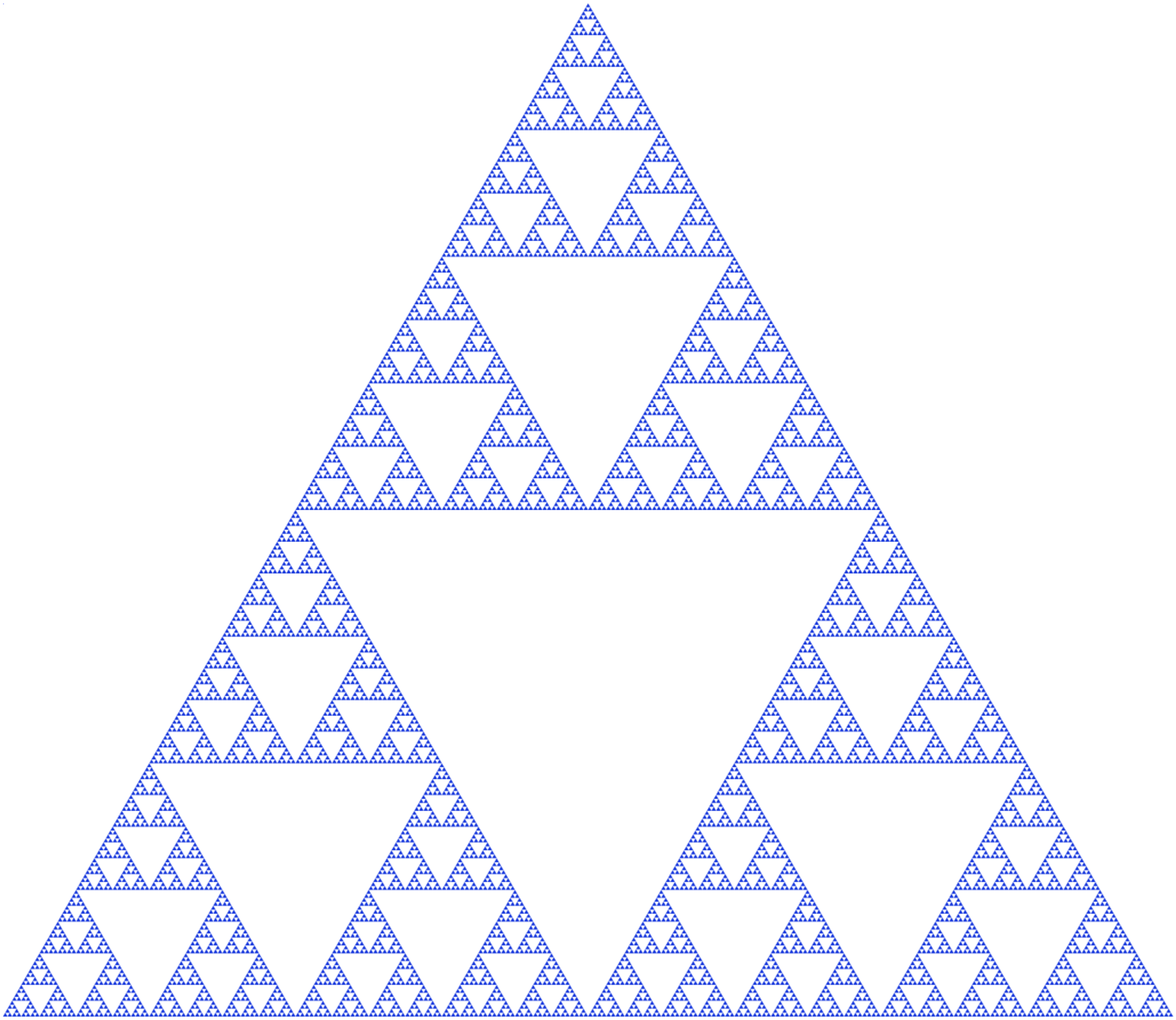

Figure 40: A fairly large stage 'n' in the construction of the Sierpinski Triangle. Image from https://commons.wikimedia.org/w/index.php?curid=8862246

Now we want to determine some properties of the Sierpinski Triangle by examining patterns in the shapes at each stage of the construction. The properties we want to determine are the area and the internal perimeter (where the **internal perimeter** means the combined perimeters of all of the figures removed from the interior of the original shape at S_0) of the Sierpinski Triangle. Fill in the table below. As you go about the work of filling in the table you will need to organize your work sufficiently well that you can later use it to communicate the thought process behind your answers.

Stage	Area	Internal Perimeter
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
⋮	⋮	⋮

1. Describe an iterative rule that can be used to generate the stage by stage construction of the Sierpinski Triangle.
2. Describe in words the patterns you notice in the numbers you calculated above. Deductively explain why your patters are generally true.
3. If the patterns in the table above continue through all the (infinite) stages of the construction of the Sierpinski Triangle, then what is the area and the internal perimeter of the Sierpinski Triangle?

Table for Organizing Relevant Information for the Construction of the Sierpinski Triangle

The table on the next page can be given to students to aid them in noticing patterns related to the stage by stage construction of the Sierpinksi Triangle. Deductive justifications should be provided for all general (stage 'n') statements.

Table 3: Table for Organizing Relevant Information for the Construction of the Sierpinski Triangle

Stage	Number of Smallest Triangles Remaining	Number of Smallest Triangles Just Removed	Area of Smallest Triangle	Side Length of Smallest Triangle	Total Area	Total Internal Perimeter
0						
1						
2						
3						
4						
5						
⋮	⋮	⋮	⋮	⋮	⋮	⋮
n						
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Test Your Understanding: The real test of your comprehension of concepts is the ability to apply them to new questions (in education circles we call this transferring the knowledge you've acquired). Use the questions below to help you determine the extent to which you have mastered the concepts and ideas we've explored above. Answers accompanied by detailed explanations are the only type of answers that count for anything.

(A) Choose a rectangle (you pick the dimensions, just make sure your have a rectangle that isn't a square, and keep in mind you may need to change your choice of dimensions). Through a process analogous to the process of creating the Sierpinski Carpet, start with your rectangle, and create a rectangular carpet. Determine the internal perimeter and area of your rectangular carpet.

(B) Below are two versions of the first few stages of the construction of fractals known as pentaflakes, which are shapes created through an interative process (like we have been studying) starting with a pentagon. The first row of the picture shows the first four stages of one way of constructing a pentaflake, and the second row shows the first four stages of a second way.

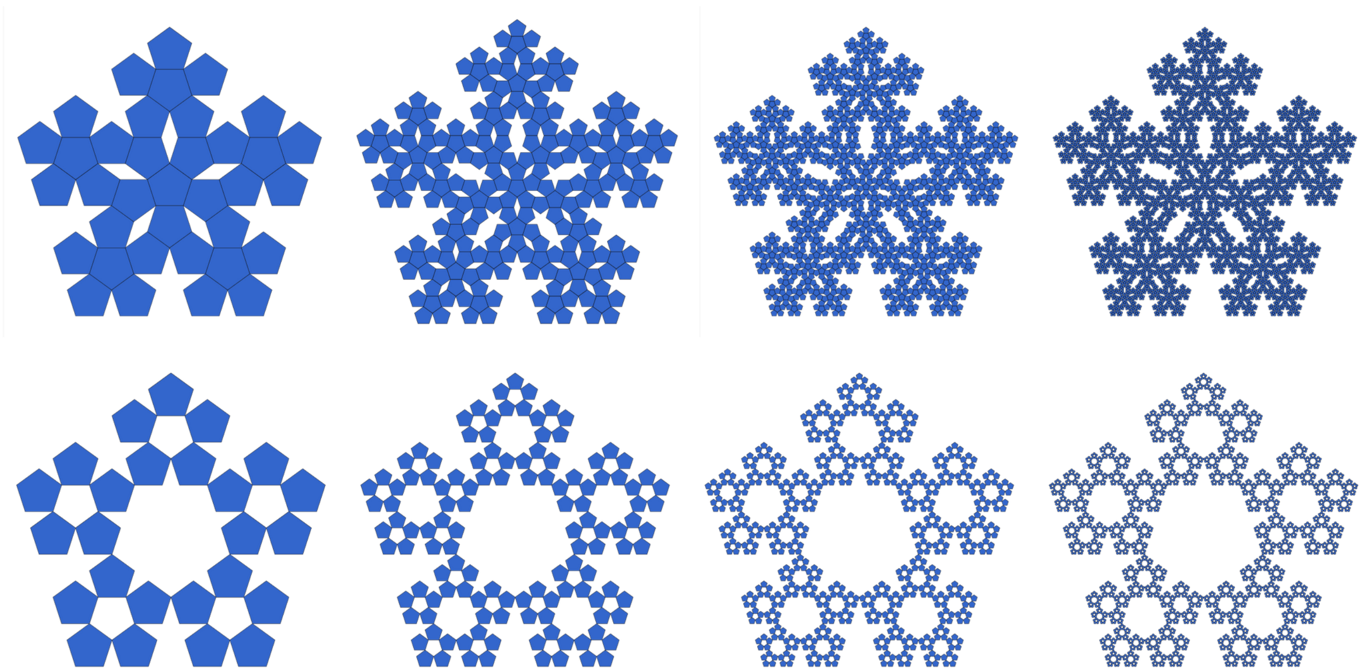

Figure 41: Two Versions of Pentaflakes. Picture from `https://en.wikipedia.org/wiki/N-flake`

Find the perimeter and area of the two versions of the pentaflake show above. (HINT: Don't try to calculate the area "taken away" at each stage, but rather try to find the scaling factor that is used at each stage to create the pieces used in the next stage.)

(C) Shown in the image below are the first four stages in the construction of a fractal object known as the Menger Sponge.

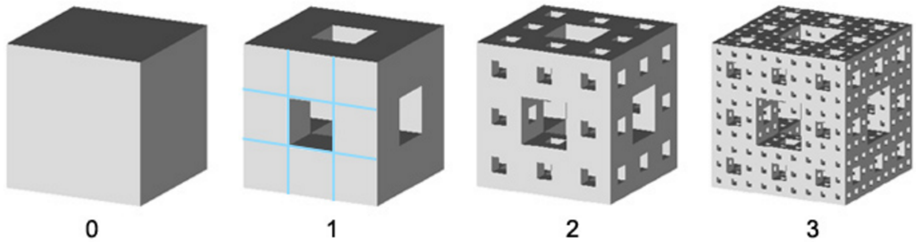

Figure 42: The first 4 stages in the construction of the fractal known as the Menger Sponge.Image from `http://fractalfoundation.org/OFC/OFC-10-3.html`

(i) If the pattern in the construction continues, find a formula for the volume of stage 'n'.

(ii) If the pattern in the construction continues, find a formula for the surface area of stage 'n'.

(ii) Use the formulas you've found to make a prediction of the volume and surface area of the Menger Sponge.

Reflect On Your Learning Experience: A fundamental skill to becoming (and being) a great teacher is to reflect on your experiences in order to learn from them and grow as an educator. Use the questions below to help guide your reflection. (NOTE: Specific answers to these questions indicate true and meaningful reflection, vague and/or nonspecific answers indicate no honest and valuable reflection.)

1. Put yourself in my position, and consider what you think my goals for you were for this exploration? What did I want you to take away from this experience?
2. What struggles did you experience during this exploration that pushed you to grow in your understanding of mathematical content?
3. What experiences did you have that helped you to better understand and appreciate the Standard of Mathematical Practice, and more generally the process of doing mathematics? About which standards did you learn the most?
4. Describe some experiences you had during this exploration with which you were impressed by how well you responded to struggle. (Examples of the sort of experiences I want you to consider could be that you found a creative way to approach a question, or that you recognized you were struggling and you actively worked past/around/through your negative feelings, or that you supported others in their work.)
5. Describe some experiences you had during this exploration where you recognize that you did not respond to struggle as well as you could. What lessons can you take away from this experience that will help you to grow into a better student, and ultimately a better educator?