Learning to Love Math through the Exploration of Maypole Patterns: Supplemental Theorems and Proofs

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ARTICLE HISTORY
Compiled January 9, 2018

1. A 5th theorem

We are grateful to a reviewer of our paper who found a fifth theorem we were missing.

Theorem 1.1 (Double Swap)**.** *The ribbon patterns of* $1A\ 2B\ 3C$ *and* $A3\ B2\ C1$ *are equivalent. Notice that leaders and followers swap ribbons but additionally, leaders 1 and 3 swap positions. Hence the name "double swap".*

1	b	1	a	1	c	1	b	1	a	1	c
a	3	c	3	b	3	a	3	c	3	b	3
2	b	2	a	2	c	2	b	2	a	2	c
a	1	c	1	b	1	a	1	c	1	b	1
3	b	3	a	3	c	3	b	3	a	3	c
a	2	c	2	b	2	a	2	c	2	b	2
1	b	1	a	1	c	1	b	1	a	1	c
a	3	c	3	b	3	a	3	c	3	b	3
2	b	2	a	2	c	2	b	2	a	2	c
a	1	c	1	b	1	a	1	c	1	b	1
3	b	3	a	3	c	3	b	3	a	3	c
a	2	c	2	b	2	a	2	c	2	b	2
1	b	1	a	1	c	1	b	1	a	1	c

Figure 1.: Screen Representation of $1A\ 2B\ 3C$

Proof. The screen representation of $A3\ B2\ C1$ is a a 90 degree turn of the original screen representation $1A\ 2B\ 3C$, as you can see in Figures 1- 2.

□

a	2	a	3	a	1	a	2	a	3	a	1
3	c	1	c	2	c	3	c	1	c	2	c
b	2	b	3	b	1	b	2	b	3	b	1
3	a	1	a	2	a	3	a	1	a	2	a
c	2	c	3	c	1	c	2	c	3	c	1
3	b	1	b	2	b	3	b	1	b	2	b
a	2	a	3	a	1	a	2	a	3	a	1
3	c	1	c	2	c	3	c	1	c	2	c
b	2	b	3	b	1	b	2	b	3	b	1
3	a	1	a	2	a	3	a	1	a	2	a
c	2	c	3	c	1	c	2	c	3	c	1
3	b	1	b	2	b	3	b	1	b	2	b

Figure 2.: Screen Representation of $A3\ B2\ C1$

2. 6 ribbons, 3 colors

We know that there are $3^6 = 729$ possibilities we need to account for. We can apply our previous knowledge because the possibilities contain scenarios in which only two colors (black and white, red and white, black and red) are used. Each of these gives us 62 combinations, see proof in our main paper.

TODO: link?

Then, we must also account for the trivial patterns $BB\ BB\ BB$, $WW\ WW\ WW$, and $RR\ RR\ RR$. All of these patterns are our "known patterns": $62 + 62 + 62 + 1 + 1 + 1 = 189$.

Since we accounted for all possibilities with one or two colors, we can now count the ones with all three colors present. We will now use color changes to make the counting easier, as opposed to the way we counted in our main paper for 6 ribbons and 2 colors. Black will be again our dominant color. We will first look at all the patterns with 4 black ribbons and 1 ribbon of each of the other colors.

Group $4 - 1 - 1$

(1) $BB\ BB\ WR$ will give us 36 possibilities: We get 6 possibilities from letter rotation, then we multiply by $1 * 2 * 3 = 6$ to account for all possible color changes. Notice that all 36 ribbon patterns will be equivalent.

(2) $BB\ BR\ BW$ will give us 36 possibilities in the same way as before.

(3) $BB\ RB\ BW$ will only give us 18 possibilities since the non-dominant color change possibilities will be the same as the letter rotation possibilities. Only the dominant color change will give us new possibilities, giving us $6 * 3 = 18$ possibilities. Notice that $BB\ BB\ WR$ can be rotated to $BB\ BW\ RB$ which is a ribbon swap of $BB\ WB\ BR$. This is a rotation of $BB\ RB\ BW$. Hence $BB\ BB\ WR$ and $BB\ RB\ BW$ have equivalent ribbon patterns.

In total, we find $36 + 36 + 18 = 90$ possibilities in the $4 - 1 - 1$ group. But there are only two non-equivalent ribbon patterns: $BB\ BB\ WR$ and $BB\ BR\ BW$.

Group $3-2-1$

(1) $BB\ BW\ WR$: 36 possibilities.

(2) $BB\ BW\ RW$: 36 possibilities.

(3) $BB\,BR\,WW$: 36 possibilities. Notice that a pair swap creates $BB\ WW\ BR$. After a ribbon swap this results in $BB\ WW\ RB$ which rotates to $BB\ BW\ WR$.

Keeping two B's together:

(4) $BB\ WW\ BR$: 36 possibilities. Notice that after a ribbon swap we get $BB\ WW\ RB$ which rotates to $BB\ BW\ WR$.

(5) $BB\ WR\ BW$: 36 possibilities. Notice that after a ribbon swap this equals $BB\ RW\ WB$ which rotates to $BB\,BR\,WW$.

(6) $BB\ RW\ BW$: 36 possibilities. Notice that after a ribbon swap this equals $BB\,WR\,WB$ which rotates to $BB\ BW\ RW$.

(7) $BB\ RB\ WW$: 36 possibilities. Notice that after a ribbon swap this equals $BB\,BR\,WW$.

(8) $BB\ WB\ RW$: 36 possibilities. Notice that after a ribbon swap we get $BB\ BW\ WR$.

(9) $BB\ WB\ WR$: 36 possibilities. Notice that a ribbon swap equals $BB\ BW\ RW$.

All B's separate:

(10) $BW\ BW\ BR$: 36 possibilities.

In total, the group $3-2-1$ has $36*10 = 360$ possibilities. There are only three non-equivalent ribbon patters: $BB\ BW\ WR$, $BB\ BW\ RW$, and $BW\ BW\ BR$.

Group $2-2-2$

We organize the possibilities by tracking where the B's are, starting with all possibilities in which 2 B's are next to each other.

(1) $BB\ WW\ RR$: Here we only need 2 letter rotations because of the structure of the letter representation. So we get 2*6=12 possibilities.

(2) $BB\ RW\ RW$: 36 possibilities. This includes $BB\,WR\,WR$.

(3) $BB\ RW\ WR$: Here we only need 3 letter rotations because of the structure of the letter representation. So we get 3*6=18 possibilities. Notice in Figure 3 that the ribbon pattern image is a 90 degree turn of $BB\ WW\ RR$ as predicted by theorem 1.1

(4) $BB\,RR\,WW$ is already accounted for as a color change of $BB\ WW\ RR$. No new possibilities.

Patterns of the form: B - two letters - B - two letters:

(5) $BW\ RB\ WR$: We only need 3 letter rotations because of the structure of the letter representation. Additionally there are only 3 more different possibilities from color changes. Hence in total there are 6 possibilities. Notice that a ribbon swap equals $WB\ BR\ RW$ which rotates to $BB\ RR\ WW$ which is a color change of $BB\,WW\ RR$.

(6) $BR\,WB\ RW$ is a color change of $BW\ RB\,WR$. No new possibilities.

(7) $BR\ RB\ WW$ can be rotated to $RR\ BW\ WB$ which is a color change of $BB\ RW\ WR$. No new possibilities.

(8) $BW\ RB\ RW$ can be rotated to $WR\ BR\ WB$ which is a color change of $BR\,WR\ BW$, coming next. No new possibilities.

Patterns of the form: B - three letters - B - one letter

(9) $BR\ WR\ BW$ Here we only need 3 letter rotations because of the structure of the letter representation. So we get 3*6=18 possibilities. Notice that after ribbon swap this is equal to $RB\ RW\ WB$ which rotates to $WW\ RB\ BR$. This is a color change of $BB\ RW\ RW$.

(10) $BW\ RR\,BW$ rotates to $RR\,BW\ BW$ which is a color change of $BB\ RW\ RW$. No new possibilities.

(11) $BR\,RW\ BW$ rotates to $RR\,WB\,WB$ which is a color change of $BB\ RW\ RW$. No new possibilities.

(12) $BR\,WW\ BR$ rotates to $WW\ BR\,BR$ which is a color change of $BB\ RW\ RW$. No new possibilities.

(13) $BW\ RW\ BR$ rotates to $WB\ RB\,WR$ which is a color change of $BR\,WR\,BW$.

(14) $BW\ WR\,BR$ rotates to $WW\ RB\ RB$ which is a color change of $BB\ RW\ RW$. No new possibilities.

In total, we found $12 + 36 + 18 + 6 + 18 = 90$ possibilities in this group but only 2 non-equivalent ribbon patterns: $BB\ WW\ RR$ and $BB\ RW\ RW$.

All three groups together account for $90 + 360 + 90 = 540$ possibilities. If we add up all possibilities from the groups and the previously computed 189 possibilities together, we find indeed $540 + 189 = 729$ possibilities.

Theorem 2.1. *There are 7 non-equivalent ribbon patterns for 6 ribbons with exactly 3 colors: $BB\ BB\ WR$, $BB\ BR\ BW$, $BB\ BW\ WR$, $BB\ BW\ RW$, $BW\ BW\ BR$,*

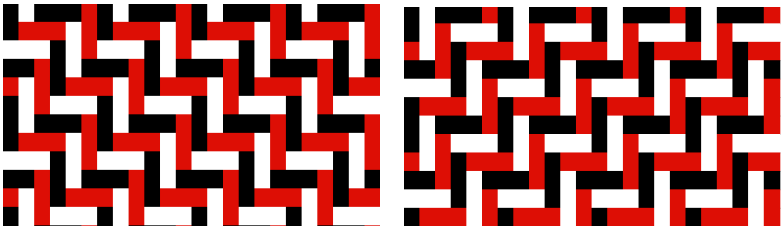

Figure 3.: Screen Representations of $BB\ WW\ RR$ and $BB\ RW\ WR$.

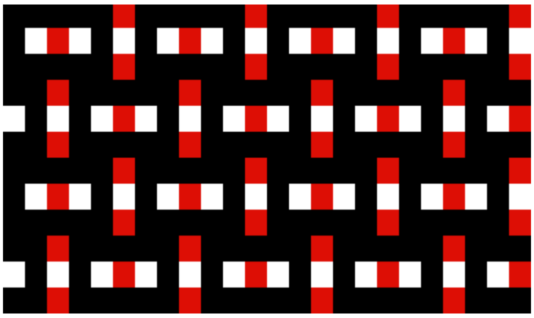

Figure 4.: Screen Representation of $BB\ BB\ WR$

$BB\ WW\ RR$, and $BB\ RW\ RW$. You can find the images in Figures 4 - 10. We looked carefully at the remaining 7 images of the ribbon patterns to ensure that they are indeed not equivalent patterns.

Proof. See the work in the list above. □

2.1. *6 Ribbons, 4 Colors*

The total number of letter representations for 6 ribbons and 4 colors is $4^6 = 4,096$. Given that we already know the results for 6 ribbons and 1, 2, and 3 colors, we will first show that there are 2,536 patterns that we can already account for:

We have 4 patterns for 6 ribbons and 1 color: $BB\ BB\ BB$, $WW\ WW\ WW$, $RR\ RR\ RR$, and $GG\ GG\ GG$.

There are 6 ways we can choose 2 out of 4 colors and each combination provides

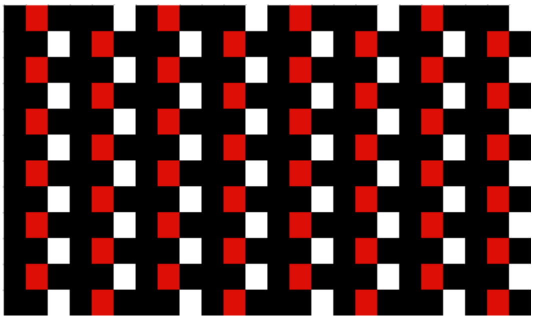

Figure 5.: Screen Representation of $BB\ BR\ BW$

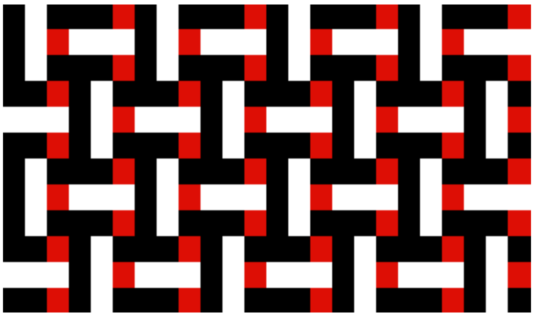

Figure 6.: Screen Representation of $BB\ BW\ WR$

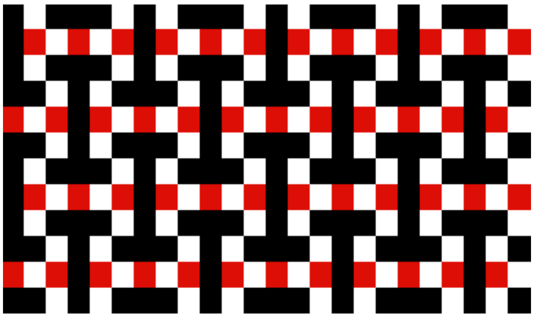

Figure 7.: Screen Representation of $BB\ BW\ RW$

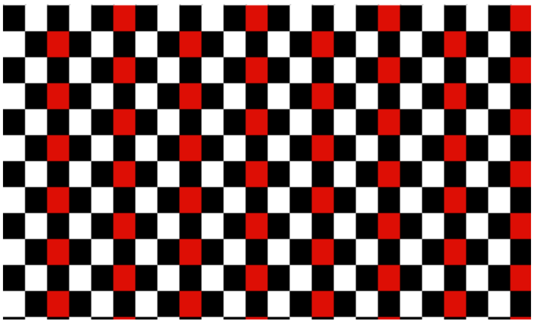

Figure 8.: Screen Representation of $BW\ BW\ BR$

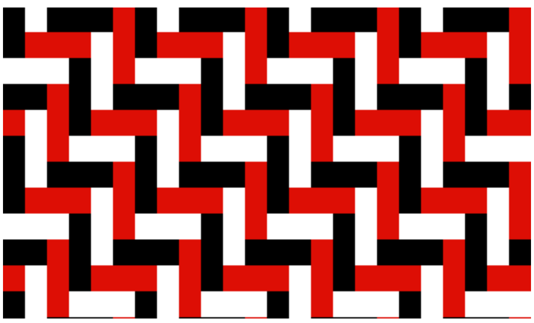

Figure 9.: Screen Representation of $BB\ WW\ RR$

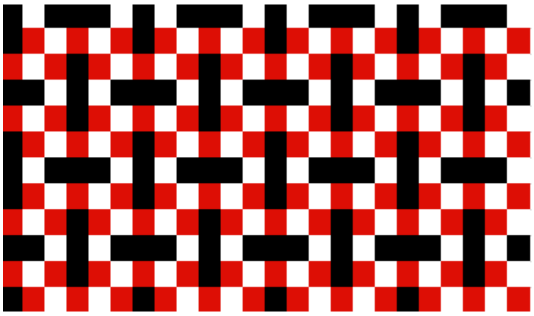

Figure 10.: Screen Representation of $BB\ RW\ RW$

us with $64 - 2 = 62$ patterns, see our main paper. Hence there are $62 \times 6 = 372$ for 6 ribbons and 2 colors already accounted for.

There are 4 ways we can choose 3 out of 4 colors, and each combination provides us with $540 = 90 + 360 + 90$ possibilities, see Section 2. Hence there are $540 \times 4 = 2,160$ for 6 ribbons and 3 colors already accounted for.

Therefore there are $4096 - 4 - 372 - 2160 = 1,560$ patterns left that we must check. We choose the black ribbon again as the dominant color, with a maximum of 3 occurrences so that all 4 colors are present in the ribbon patterns. We will again order our argument by checking several groups of patterns:

Group $3 - 1 - 1 - 1$

We have 3 black ribbons and 1 green, 1 white and one red ribbon.

(1) $BB\ BR\ GW$ we have 6 possibilities for the letter rotations and then have to multiply by $2 \times 3 \times 4 = 24$ for the color changes. Hence there are $6 \times 24 = 144$ possibilities.

(2) $BB\ RG\ BW$: 144 possibilities as above. Notice that this is a ribbon swap of $BB\ GR\ WB$ which rotates to $BB\ BG\ RW$ which is a color change of $BB\ BR\ GW$.

(3) $BB\ RB\ GW$: 144 possibilities. Notice that this is a ribbon swap of $BB\ BR\ WG$ which is a color change of $BB\ BR\ GW$.

(4) $BG\ BR\ BW$: Because of the structure of the letter representation we only need 2 letter rotations, giving us 2*24=48 possibilities.

Group $2 - 2 - 1 - 1$

We have 2 black and 2 red ribbons and 1 green and 1 white ribbon.

B's together and R's together:

(1) $BB\ RR\ GW$: 144 possibilities.

(2) $BB\ GR\ RW$ Here we need only 3 letter rotations because of the structure of the letter representation. Multiplied by 24 this leads to 72 possibilities.

(3) $BB\ GW\ RR$ is already included as a rotation and color change of $BB\ RR\ GW$

B's together and R's apart:

(4) $BB\ RG\ RW$: 144 possibilities.
(5) $BB\ GR\ WR$: 144 possibilities. Notice that this is a ribbon swap of $BB\ RG\ RW$.

(6) $BB\ RG\ WR$: 144 possibilities. Notice that this is a ribbon swap of $BB\ GR\ RW$.

B's apart and R's apart. The B's are one apart.

(7) $BG\ BR\ WR$: Here we need only 3 letter rotations because of the structure of the letter representation. Multiplied by 24 this leads to 72 possibilities. Notice that a leader swap will change this into $WG\ BR\ BR$ which gets rotated to $BR\ BR\ WG$. This is a color change of $BR\ BR\ GW$, see next.
(8) $BR\ BR\ GW$: 144 possibilities.

(9) $BR\ BG\ RW$: 144 possibilities.Notice that when we exchange red again black we get $RB\ RG\ BW$ which rotates to $BR\ GB\ WR$. A leader rotation gives $WR\ BB\ GR$ and this rotates to $BB\ GR\ WR$. This is a ribbon swap of $BB\ RG\ RW$.

(10) $BRBGWR$: this is a rotation of $RB\ RB\ GW$ which is a color change of $BR\ BR\ GW$. No new possibilities.

B's apart and R's apart. The B's are two apart.

(11) $BR\ GB\ RW$: Here we need only 3 letter rotations because of the structure of the letter representation. Multiplied by 24 this leads to 72 possibilities. Notice that a ribbon swap will lead to $RB\ BG\ WR$ which will rotate to $RR\ BB\ GW$ which is a color change of $BB\ RR\ GW$.

(12) $BR\ GB\ WR$. This is a rotation of $RB\ RG\ BW$ which is a color change of $BR\ BG\ RW$ and already accounted for.

(13) $BG\ RB\ WR$: this is a rotation of $RB\ GR\ BW$ and thus a color change of $BR\ GB\ RW$. There are no new possibilities.

In Figure 11, notice that $BB\ RR\ GW$ is a 90 degree turn and a color change (red against black and green against white) of $BB\ GR\ RW$ as predicted by theorem 1.1.

Theorem 2.2. *For 6 ribbons and 4 colors there are 5 non-equivalent ribbon patterns: $BB\ BR\ GW$, $BG\ BR\ BW$, $BB\ RR\ GW$, $BB\ RG\ RW$, and $BR\ BR\ GW$. You can see them in Figures 12 - 16.*

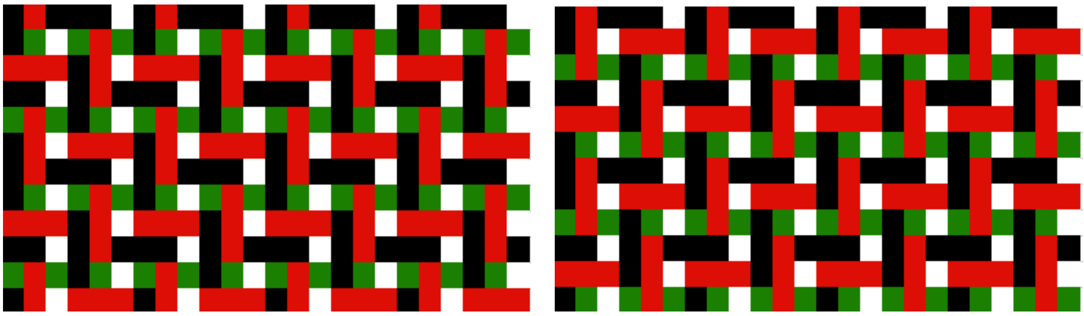

Figure 11.: Screen Representations of *BB RR GW* and *BB GR RW*.

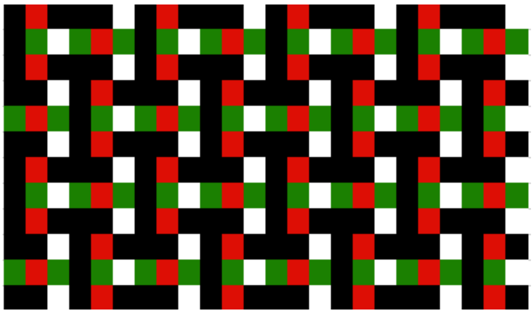

Figure 12.: Screen Representation of *BB BR GW*

Proof. See work in list above. We looked carefully at the remaining 5 images of the ribbon patterns to ensure that they are indeed not equivalent patterns. □

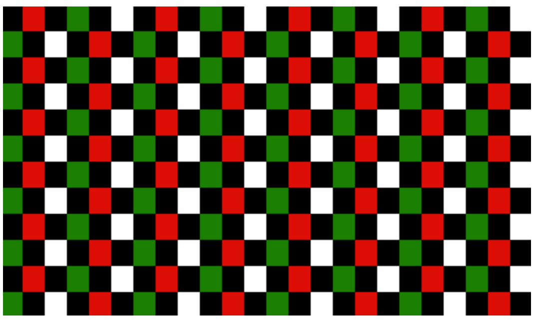

Figure 13.: Screen Representation of *BG BR BW*

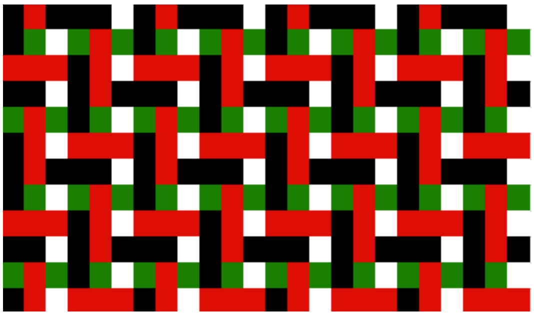

Figure 14.: Screen Representation of *BB RR GW*

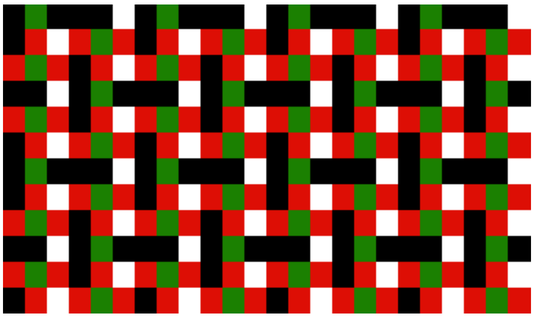

Figure 15.: Screen Representation of *BB RG RW*

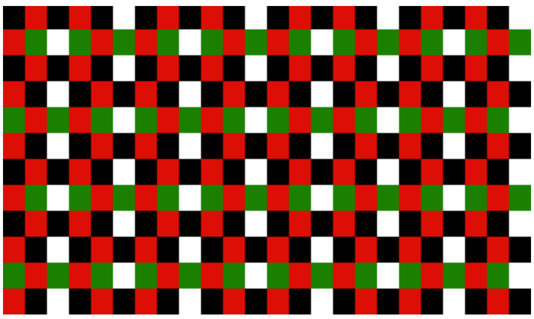

Figure 16.: Screen Representation of *BR BR GW*