Exploring Operations

Figure 16: The Arecibo Radio Telescope, at Arecibo, Puerto Rico. At 1000 feet (305 m) across, it is the second largest dish antenna in the world. The dish, built into a bowl in the landscape, focuses radio waves from the sky on the feed antenna suspended above it on cables. Since the dish itself can't move, the telescope is steered to point at different regions of the sky by moving the feed antenna (dome) along the curving metal track. It is used in three major areas of research: radio astronomy, atmospheric science, and radar astronomy. Scientists who want to use the observatory submit proposals that are evaluated by an independent scientific board. The observatory has appeared in film and television productions, gaining more recognition in 1999 when it began to collect data for the SETI@home project. Picture from `https://commons.wikimedia.org/w/index.php?curid=16005203`

Suppose that at some point in the not-so-distant future a signal from deep space is received on a SETI (`http://www.seti.org`) radio telescope. The signal is taken to the best cryptologists who apply their most sophisticated algorithms to decipher the meaning of the signal. The algorithms these cryptologists used come from a branch of mathematics known as coding theory. When the signal was finally understood it was because an ingenuitive young mathematician noticed that this signal was based in a mathematics built from an alternative type of arithmetic. In this alien arithmetic there are two basic operations, non-terrestrial addition (which we will denote by $\oplus$), and non-terrestrial multiplication (which we will denote by $\otimes$). The true insight was that the non-terrestrial addition and non-terrestrial multiplication are very different than our Earthly addition and multiplication. Even though they are different, the universality of mathematics allows us to translate between the two. Indeed we realized that if x and y are real numbers, then

$$x \oplus y = \min(x, y) \qquad \text{and} \qquad x \otimes y = x + y,$$

where $\min(x, y)$ means the minimum of x and y, and to be clear, $+$ is our Earthly addition.

1. Find the following

(a) $3 \oplus 7$,

(b) $3 \otimes 7$.

2. Recall what the commutative property is for Earthly addition and multiplication. Demonstrate that these processes are commutative using a model of this mathematical relationship. (A model can be picture/diagram/manipulative/other thing that helps make an abstract mathematical relationship more concrete in such a way that will facilitate a deeper conceptual understanding of that piece of mathematics.)

3. Are the non-terrestrial operations of addition and multiplication commutative? Give inductive and deductive arguments that support your conclusion.

4. Recall what the distributive property is for Earthly addition and multiplication. Demonstrate that the distributive law is valid using a model of this mathematical relationship. (A model can be picture/diagram/manipulative/other thing that helps make an abstract mathematical relationship more concrete in such a way that will facilitate a deeper conceptual understanding of that piece of mathematics.)

5. Do the non-terrestrial operations of addition and multiplication abide by the distributive law? Give inductive and deductive arguments that support your conclusion.

6. Recall what the associative property is for Earthly addition and multiplication. Demonstrate that these processes are associative using a model of this mathematical relationship. (A model can be picture/diagram/manipulative/other thing that helps make an abstract mathematical relationship more concrete in such a way that will facilitate a deeper conceptual understanding of that piece of mathematics.)

7. Are the non-terrestrial operations of addition and multiplication associative? Give inductive and deductive arguments that support your conclusion.

8. It turns out that a strange property of non-terrestrial arithmetic is what might be considered a dream to many students, namely that exponents interact with non-terrestrial arithmetic in the following way:

   For any whole number n and any real numbers x and y we have that $(x \oplus y)^n = x^n \oplus y^n$.

   Explain why this is true. Is this true of how exponents interact with Earthly arithmetic? Thus is it true that

   For any whole number n and any real numbers x and y we have that $(x + y)^n = x^n + y^n$.

   Explain all of your conclusions with both inductive and deductive arguments. (Be sure to explain what we mean by an exponent.)

9. Fill out the following table for non-terrestrial addition table:

$\oplus$	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												
13												

(HINT: Don't just try to fill out all 156 different entries in the table. Find patterns in how $\oplus$ works.) Find many general patterns, in the same way that we have elementary school students find patterns in Earthly addition tables. And give deductive explanations for why those patterns are generally true.

10. Fill out the following table for non-terrestrial multiplication table:

$\otimes$	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												
13												

(HINT: Don't just try to fill out all 156 different entries in the table. Find patterns in how $\oplus$ works.) Find many general patterns, in the same way that we have elementary school students find patterns in Earthly addition tables. And give deductive explanations for why those patterns are generally true.

11. In Earthly addition and multiplication we have an additive identity (zero), and a multiplicative identity (one). Are there non-terrestrial additive and multiplicative identities? Explain.

12. Do non-terrestrial addition and non-terrestrial multiplication have inverse operations? Is non-terrestrial subtraction possible? Explain. Is non-terrestrial division possible? Explain.

13. Up until this point we haven't yet actually stated a definition for what we mean when we talk about an "operation" on numbers. Of course from elementary school we have the familiar "operations" of addition (+), subtraction (−), multiplication (×), and division (÷), and we just explored the "operations" of alien addition ($\oplus$), alien multiplication ($\otimes$), and the existence (or not) of their inverses. Using all of your experience with $+, -, \times, \div, \oplus,$, and $\otimes$ work as a group to define the term "**operation**" in a mathematical context. Be ready to share your definition with the class.

Test Your Understanding: The real test of your comprehension of concepts is the ability to apply them to new questions (in education circles we call this transferring the knowledge you've acquired). Use the questions below to help you determine the extent to which you have mastered the concepts and ideas we've explored above. Answers accompanied by detailed explanations are the only type of answers that count for anything. (NOTE: The formulas below use the standard operations.)

(A) Determine whether or not the following operations are commutative and associative. In each question the symbol for the operation is $\bowtie$, and the definition of the operation (different in each part) is provided through the given equation.

1. For any real numbers a and b, $a \bowtie b = 7$,
2. For any real numbers a and b, $a \bowtie b = a - b$ (usual subtraction),
3. For any real numbers a and b, $a \bowtie b = \max(a, b) + 5$ (where $\max(a, b)$ is the maximum of a and b),
4. For any real numbers a and b, $a \bowtie b = \max(a, b) - 9$ (where $\max(a, b)$ is the maximum of a and b),
5. For any real numbers a and b, $a \bowtie b = 8a + 8b + 3$,
6. For any real numbers a and b, $a \bowtie b = a^3 + b^7$.
7. For any natural numbers (positive whole numbers) a and b, $a \bowtie b = a - b$ (usual subtraction),
8. For any natural numbers (positive whole numbers) a and b, $a \bowtie b = ab$
9. For any integers (positive and negative whole numbers) a and b, $a \bowtie b = \frac{a}{b}$
10. For any integers a and b, $a \bowtie b = \frac{a}{6} + \frac{7}{b}$

(B) Suppose that after examining the distributive property in class one of your students said the following to you:

> The distributive property of addition (+) and multiplication ($\times$) when we look at positive and negative whole numbers just follows from the fact that multiplication is defined as repeated addition.

Is this student correct? How would you respond to the student? What if the student had said this was true for all numbers, not just the integers, is she or he correct now? How would you respond?

(C) Does multiplication ($\times$) distribute over subtraction ($-$)? Explain.

(D) Take the operations from (A) and determine whether these operations have identities and inverse operations.

(E) Develop your own operation (different enough from the ones above that it is not just tweaking some numbers) that satisfies the conditions:

(i) Has an identity, but no inverse operation.

(ii) Has an identity, and has an inverse operation.

(F) P.E.M.D.A.S. is an acronym for Parentheses, Exponents, Multiplication, Division, Addition and Subtraction, and is used to help student remember the order in which we unravel an expression which involves combinations of these notations. Mathematicians call this order of operations a **convention**, which means it is an agreement that if everyone follows this order when unraveling mathematical expressions, then we will all agree on their value.

- How would you explain the need for a convention to an elementary school student who is familiar with all of the notations involved in PEMDAS?
- When the numbers involved are integers, connect all of the operations involved in PEMDAS to addition, and then explain how PEMDAS can be viewed as a natural consequence of repeated addition.

Reflect On Your Learning Experience: A fundamental skill to becoming (and being) a great teacher is to reflect on your experiences in order to learn from them and grow as an educator. Use the questions below to help guide your reflection. (NOTE: Specific answers to these questions indicate true and meaningful reflection, vague and/or nonspecific answers indicate no honest and valuable reflection.)

1. Put yourself in my position, and consider what you think my goals for you were for this exploration? What did I want you to take away from this experience?
2. What struggles did you experience during this exploration that pushed you to grow in your understanding of mathematical content?
3. What experiences did you have that helped you to better understand and appreciate the Standard of Mathematical Practice, and more generally the process of doing mathematics? About which standards did you learn the most?
4. Describe some experiences you had during this exploration with which you were impressed by how well you responded to struggle. (Examples of the sort of experiences I want you to consider could be that you found a creative way to approach a question, or that you recognized you were struggling and you actively worked past/around/through your negative feelings, or that you supported others in their work.)
5. Describe some experiences you had during this exploration where you recognize that you did not respond to struggle as well as you could. What lessons can you take away from this experience that will help you to grow into a better student, and ultimately a better educator?