Divisibility Rules I: Base 10 Number System

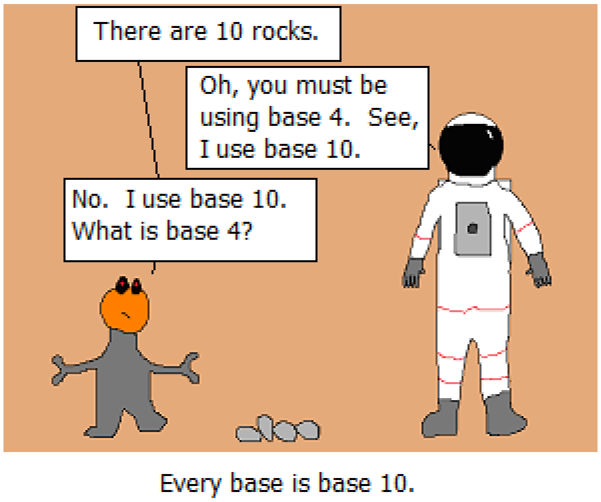

Figure 9: HINT (for the joke): What is the number symbol for the amount of dots here • • •• in a base 4 number system. After you think about this, if you don't get the joke, please see me before the next exploration. Picture from `http://math.stackexchange.com/questions/166869/is-10-a-magical-number-or-i-am-missing-something`

For The Purposes Of This Exploration When We Talk About A Number We Mean A Positive Whole Number

1. Start by finding patterns in the divisibility of multiples of 10 by the single digit numbers.

    - What pattern is there when you divide the multiples of 10 by 1? Explain why your pattern is generally true.
    - What pattern is there when you divide the multiples of 10 by 2? Explain why your pattern is generally true.
    - What pattern is there when you divide the multiples of 10 by 3? Explain why your pattern is generally true.
    - What pattern is there when you divide the multiples of 10 by 4? Explain why your pattern is generally true.
    - What pattern is there when you divide the multiples of 10 by 5? Explain why your pattern is generally true.
    - What pattern is there when you divide the multiples of 10 by 6? Explain why your pattern is generally true.
    - What pattern is there when you divide the multiples of 10 by 7? Explain why your pattern is generally true.
    - What pattern is there when you divide the multiples of 10 by 8? Explain why your pattern is generally true.

- What pattern is there when you divide the multiples of 10 by 9? Explain why your pattern is generally true.

Having complete and detailed explanations to the questions in (1) is extremely important, because we will use them in the rest of the exploration.

2. Use your results from the last question to determine, and then explain, the pattern for dividing powers of 10 (i.e. 10^1, 10^2, 10^3, ...) by the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ respectively.

3. If you spend enough time out on the mean streets of any major city you'll hear a lot of things. Some of these things are true, and some are not. The last time I was out on the mean streets I heard a rule about dividing a number by 9. The rule I heard was

   > A positive whole number is divisible by 9 if the number equal to the sum of the digits is divisible by 9.

   Is this rule true (in general)? If you think this is true then explain why deductively. (Hint: What do the digits of a number represent?)

4. The last time that I was standing in line in a grocery store I saw a tabloid whose cover story was a rumor about a rule that celebrities use to determine whether a number is divisible by 3. The rule in the story was

   > A positive whole number is divisible by 3 if the number equal to the sum of the digits is divisible by 3.

   Is this rule true (in general)? If you think this is true then explain why deductively. (Hint: Again, what do the digits of a number represent?)

5. Hearing and reading about these divisibility rules I started doing my own research and I found the following rules on the internet:

   - A positive whole number is divisible by 2 if the ones digit of the number is even (i.e. a multiple of 2).
   - A positive whole number is divisible by 5 if the ones digit of the number is either 0 or 5.
   - A positive whole number is divisible by 10 if the ones digit is 0.

   The website on which I found these rules didn't have any explanations about why they are true, so I'm skeptical (don't believe everything you read online), are these rules true (in general)? If you think each is true then explain why deductively.

6. What number(s) x, if any, can you find a divisibility rule for that will be of the form

   > A positive whole number is divisible by x if the number formed by the tens digit and the ones digit of that number is divisible by x.

   Explain. (NOTE: When I say "he number formed by the tens digit and the ones digit of that number" I mean the number you get by only looking at those digits, for example if we start with the number 2014 then the number formed by the tens digit and the ones digit is 14.)

7. What number(s) x, if any, can you find a divisibility rule for that will be of the form

   > A positive whole number is divisible by x if the number formed by the hundreds digit, the tens digit, and the ones digit of that number is divisible by x.

   Explain. (NOTE: When I say "the number formed by the hundreds digit, the tens digit, and the ones digit of that number " I mean the number you get by only looking at those digits, for example if we start with the number 51987 then the number formed by the hundreds digit, the tens digit, and the ones digit is 987.)

8. What number(s) x, if any, can you find a divisibility rule for that will be of the form

   > A positive whole number is divisible by x if the number formed by taking the sum of twice the tens digit and the ones digit of that number is divisible by x.

   Explain. (NOTE: When I say "the number formed by taking the sum of twice the tens digit and the ones digit of that number" I mean, for example, if we start with the number 2014 then we take twice the tens digit, so $2 \times 1 = 2$, and the ones digit 4, and add them together to get $2 + 4 = 6$.)

9. What number(s) x, if any, can you find a divisibility rule for that will be of the form

   > A positive whole number is divisible by x if the number formed by taking the sum of four times the hundreds digit, twice the tens digit, and the ones digit of that number is divisible by x.

   Explain. (NOTE: When I say "the number formed by taking the sum of four times the hundreds digit, twice the tens digit, and the ones digit " I mean, for example, if we start with the number 51987 then we take four times the hundreds digit, so $4 \times 9 = 36$, twice the tens digit, so $2 \times 8 = 16$, and the ones digit 7, and add them together to get $36 + 18 + 7 = 61$.)

10. Find a divisibility rule for 6, and explain why your rule is generally true.

Test Your Understanding: The real test of your comprehension of concepts is the ability to apply them to new questions (in education circles we call this transferring the knowledge you've acquired). Use the questions below to help you determine the extent to which you have mastered the concepts and ideas we've explored above. Answers accompanied by detailed explanations are the only type of answers that count for anything.

(A) What is special about the numbers 2, 5, and 10 that allows us to determine whether a number (which could potentially have a lot of digits) is divisible by them by focusing on only the ones digit? (Isn't it crazy that we know that the number

$$35213576829872093809287409827409709224729570297275019752093520852985727209572572905419430416943643041 8$$

is divisible by 2, and is not divisible by 5, or by 10 by only looking at one of the one hundred and one digits in the number?)

(B) Find a divisibility rule for 7, and explain why your rule is generally true.

(C) Find a divisibility rule for 11, and explain why your rule is generally true.

(D) What number(s) x, if any, can you find a divisibility rule for that will be of the form

> A positive whole number is divisible by x if the number formed by the ten-thousands digit, the thousands digit, hundreds digit, the tens digit, and the ones digit of that number is divisible by x.

Explain your conclusion with deductive reasoning. (NOTE: When I say "the number formed by the ten-thousands digit, the thousands digit, hundreds digit, the tens digit, and the ones digit of that number " I mean the number you get by only looking at those digits, for example if we start with the number 36251987 then the number formed by the ten-thousands digit, the thousands digit, hundreds digit, the tens digit, and the ones digit is 51987.)

(E) Determine if the following divisibility rule is generally valid. If you determine that it is not generally valid then find a number that it fails to account for, but if you determine that it is generally valid, then explain why with deductive reasoning.

> A positive whole number is divisible by 37 if the number formed by adding the digits in blocks of three from right to left is divisible by 37.

(NOTE: When I say "the number formed by adding the digits in blocks of three from right to left" I mean the number you get by breaking the number up and adding as in the following examples, $653, 272$ is divisible by 37 because $653 + 272 = 925$, and 925 is divisible by 37.)

(F) Consider dividing numbers by 11.

(i) Notice that $1 \div 11 = 0R1$, $100 \div 11 = 9R1$, $10000 \div 11 = 909R1$, $1000000 \div 11 = 90909R1$. Explain why in general it will be true that every even power of 10 (i.e. 10^0, 10^2, 10^4, 10^6, 10^8,...) will have a remainder of 1 when divided by 11.

(iI) For the odd powers of 10 consider that we can think of $10 \div 11 = 0R10$ as the same thing as $10 \div 11 = 1R - 1$, $1000 \div 11 = 90R10 = 91R - 1$, $100000 \div 11 = 9090R10 = 90901R - 1$. Explain why in general it will be true that every odd power of 10 (i.e. 10^1, 10^3, 10^5, 10^7, 10^9,...) will have a remainder of 10 when divided by 11, and we can think of this as the same things as a remainder of -1 if the total number of groups of eleven we have is one more than if the remainder is 10.

(iii) Use what you've shown in the first two parts of this question to explain why the following divisibility rule for 11 is generally true.

> If the alternating sum of the digits of a positive integer is divisible by 11, then the original number is divisible by 11.

Explain your conclusion with deductive reasoning. (NOTE: When I say "the alternating sum of the digits of a positive integer " I mean the number you get by adding the digits where you alternate the sign, for example if we start with the number 918,082 then since $9-1+8-0+8-2=22$ which is divisible by 11, then the original number is also divisible by 11 ($918,082 \div 11 = 83,462$).

(G) Find a divisibility rule for 1024, and explain why your rule is generally true.

(H) Consider the rule below for divisibility by 7 (which should be different than the rule you found in a previous question). Explain why this rule holds true.

> To find out if a number is divisible by 7, take the last digit, double it, and subtract it from the rest of the number. If you get an answer divisible by 7 (including zero), then the original number is divisible by seven. (NOTE: If you don't know the new number's divisibility, you can apply the rule repeatedly until you can make a determination.)

(I) We have seen two general examples of types of divisibility rules. One type has been of the sort where you can determine if a number's divisibility by just considering the divisibility of a number formed by some of the original numbers digits. (You explored this type of divisibility rule in questions 4, 5, and 6 of the exploration.) A second type has been of the sort you explored where you can determine a number's divisibility by doing a calculation with the digits of the number, and look at the divisibility of the number you get out of that calculation. (You explored this type of divisibility rule in questions 3,4,8, and 9 of the exploration from class.) For this question consider representing numbers in a base 6 number system, and find one divisibility rule of each of these types.

Reflect On Your Learning Experience: A fundamental skill to becoming (and being) a great teacher is to reflect on your experiences in order to learn from them and grow as an educator. Use the questions below to help guide your reflection. (NOTE: Specific answers to these questions indicate true and meaningful reflection, vague and/or nonspecific answers indicate no honest and valuable reflection.)

1. Put yourself in my position, and consider what you think my goals for you were for this exploration? What did I want you to take away from this experience?
2. What struggles did you experience during this exploration that pushed you to grow in your understanding of mathematical content?
3. What experiences did you have that helped you to better understand and appreciate the Standard of Mathematical Practice, and more generally the process of doing mathematics? About which standards did you learn the most?
4. Describe some experiences you had during this exploration with which you were impressed by how well you responded to struggle. (Examples of the sort of experiences I want you to consider could be that you found a creative way to approach a question, or that you recognized you were struggling and you actively worked past/around/through your negative feelings, or that you supported others in their work.)
5. Describe some experiences you had during this exploration where you recognize that you did not respond to struggle as well as you could. What lessons can you take away from this experience that will help you to grow into a better student, and ultimately a better educator?