Inquiry-Based Calculus III

Mairead Greene, Amy Ksir, Christine von Renesse

January 4, 2015

Chapter 1

Introduction

Welcome to calculus in three dimensions!

In this course you will take the ideas and facts from Calculus I and II, and extend them to our three-dimensional world. You should expect to get the following out of this course.

Object Typing. Calculus III expands our ability to model the real world and analyze quantitative information, by taking the tools from Calculus I and II and extending them to situations where the input or output of a function may be a vector, rather than just one number. Along with this comes a proliferation of different types of mathematical objects we will be working with: numbers, functions, vectors, and vector fields; lines, curves, planes, surfaces, regions of the plane, and regions of three-dimensional space. One of the habits of mind you will develop in this course is of keeping track which formulas and equations are describing which types of objects. It is similar to the habit that you develop in science, and engineering courses of keeping track of the units of quantities. By the end of this course, you should expect to be able to do the following, with objects of the types listed above.

- Given an equation or formula in context, recognize what type of object it is describing.
- Given an object type, create examples of equations or formulas describing an object of that type.
- Develop the habit of noticing object type when reading and writing formulas and equations.

3D visualization. When working with functions with multiple inputs and/or outputs, it is also very helpful to be good at picturing, describing, and drawing objects in three dimensions. By the end of this course, you should expect to be able to graph and analyze curves and surfaces in three dimensions.

Computation, understanding and interpretation of different types of derivatives. In this course there will be six different types of derivatives (!): derivatives of vectors, partial derivatives, directional derivatives, gradient, divergence, and curl. The actual computations just involve putting together skills you already have from Calculus I and II; you should expect by the end of this course to be fluent in these computations. In addition you should understand for each its graphical interpretation, object type, and properties. In addition, you should expect to develop a sense of what each represents in modeling situations: what are its units? What does it tell you?

Computation, understanding and interpretation of different types of integrals. There will also be five different types of integrals: double, triple, line, surface, and flux. These generally require more geometric understanding in addition to the integration skills you learned in Calculus II. By the end of the course, you should expect to be fluent in all five, and to understand their graphical interpretations. And again, you should expect to develop a sense of what each represents in applications outside of mathematics.

Fundamental Theorems of Calculus. At the end of the course you will learn a series of powerful theorems which extend the Fundamental Theorem of Calculus to higher dimensions. These theorems are important in understanding electricity and magnetism, the flow of air and water, heat transfer, and many other physics and engineering applications. You should expect to understand what the theorems say, how they are related to each other, and how to use them.

Mathematical reasoning and logic. In all human endeavors, we make conjectures about how the world works. Different academic disciplines have different ways of determining whether a conjecture is true or false. In the sciences, one tests a theory using experiments and observations. Mathematics works a bit differently: we start with a set of basic assumptions and precise definitions, then use deductive logic to determine what mathematical rules must be true based on those assumptions. By the end of this course you should expect to have improved your ability to follow and to create such mathematical arguments.

Oral and written communication. A brilliant idea, poorly communicated, may be lost to the world forever. Being able to write and communicate clearly is one of the most important skills for a junior officer. By the end of this course, you should expect to be able to clearly explain a mathematical argument or computation, both speaking to an audience and in writing.

Technical confidence. By the end of this course, you will know more math than you did at the beginning. You should expect to feel that way!

In Chapter 2, we will begin by delving into the geometry of three dimensions, which we will use throughout the rest of the course. In Chapter 3, we will take derivatives in higher dimensions; in Chapter 4 we will do integrals. In Chapter 5 we will finish up with the fundamental theorems of calculus that link together all of the ideas of the semester.

Let us begin!

Chapter 2

Curves and Surfaces

Definition 1. *(Not really a definition). The **dimension** of a geometric object is the number of degrees of freedom on the object, or the number of numbers you need to determine a point on the object.*

Definition 2. *If a geometric object X is a subset of* $\mathbb{R}^n$*, and has dimension* $\dim(X)$*, then the **codimension** of X is* $n - \dim(X)$*.*

Problem 1. *Sketch the graph of the line* $L = \{(x,y) \in \mathbb{R}^2 : y = 3x+1\}$.

Problem 2.

1. *What is the dimension of the line L in Problem 1?*
2. *What is the codimension of L?*

Problem 3. *Which of the following points are on the line L, and which are not? Explain your reasoning.*

1. $(10,31)$
2. $(1,2)$
3. $(0,0)$
4. $(0,1)$

Problem 4. *Give five examples of points that are on the line L that are not listed in Problem 3.*

Problem 5. *For each of the points in Problem 3 that is on L, find another line so that the point is the intersection of L with this line.*

Problem 6. *For the points in Problem 3 that are on L, can you find one curve that intersects L in all of those points, but nowhere else?*

Problem 7. *Let* $G : \mathbb{R} \to \mathbb{R}^2$ *be the function defined by* $G(t) = (1+t, 4+3t)$*. On one graph, plot* $G(-2)$*,* $G(-1)$*,* $G(0)$*,* $G(1)$*, and* $G(2)$*.*

Problem 8. *Prove or find a counterexample: If a is a real number, then* $G(a)$ *is on L.*

Problem 9. *Prove or find a counterexample: If Q is a point on L, then there is a number a such that* $G(a) = Q$*.*

Problem 10. *Let $H : \mathbb{R} \to \mathbb{R}^2$ be the function defined by $H(t) = (-t, 1-3t)$. On one graph, plot $H(-2)$, $H(-1)$, $H(0)$, $H(1)$, and $H(2)$.*

Problem 11. *Compare and contrast G and H.*

Problem 12. *Suppose that $G(t)$ represents the position of a giraffe t hours after noon, and $H(t)$ represents the position of a hippopotamus t hours after noon.*

1. *What is the speed of the giraffe? What is the velocity of the giraffe? What is the speed of the hippo? What is the velocity of the hippo?*

2. *Suppose that a cheetah is going down the same road as the giraffe, but three times as fast. Write a formula for a function $C : \mathbb{R} \to \mathbb{R}^2$ so that $C(t)$ represents the position of the cheetah t hours after noon.*

Problem 13. *Prove that the range of H is L. In other words, repeat problems 8 and 9 for H.*

Problem 14. *Sketch and describe the graph of $P = \{(x,y,z) \in \mathbb{R}^3 : y = 3x+1\}$.*

Problem 15. 1. *What is the dimension of P in Problem 14?*

2. *What is the codimension of P?*

Problem 16. *Which of the following points are on P, and which are not? Explain your reasoning.*

1. $(10,31,0)$

2. $(10,31,53)$

3. $(1,1,1)$

4. $(0,0,0)$

5. $(0,1,0)$

6. $(0,1,12)$

Problem 17. *Let $f : \mathbb{R} \to \mathbb{R}^3$ be the function defined by $f(t) = (t, 1+3t, t)$. On one graph, plot $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.*

Problem 18. *Prove or find a counterexample: If a is a real number, then $f(a)$ is on P.*

Problem 19. *Prove or find a counterexample: If Q is a point on P, then there is a number a such that $f(a) = Q$.*

Problem 20. *Let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by $S(u,v) = (1+u, 4+3u, v)$.*

1. *Graph the four points $S(-2,0)$, $S(2,0)$, $S(2,3)$, and $S(-2,3)$.*

2. *Now graph the curves $S(-2,v)$ and $S(2,v)$ for $0 \le v \le 3$.*

3. *Now graph the curves $S(u,0)$ and $S(u,3)$ for $-2 \le u \le 2$.*

4. *Describe the surface.*

Problem 21. *Summarize the relationship between the sets and functions in Problems 1, 7, 14, 17, and 20.*

Problem 22. *What is the range of f?*

Problem 23. *What is the range of S?*

Problem 24. *Let R be the rectangle $[-2,2] \times [-2,2]$, and let $T : R \to \mathbb{R}^3$ be the function defined by $T(u,v) = (u,v,4-u^2)$.*

1. *Graph the four points $T(-2,-2)$, $T(-2,2)$, $T(2,2)$, and $T(2,-2)$.*
2. *Now graph the curves $T(-2,v)$ and $T(2,v)$ for $-2 \le v \le 2$.*
3. *Now graph the curves $T(u,-2)$ and $T(u,2)$ for $-2 \le u \le 2$.*
4. *Describe the surface.*

Let D be a set of points in the plane $\mathbb{R}^2$. Let P be a point in the plane.

P is an *interior* point of D if and only if there is a (possibly very tiny!) positive real number r, so that the open disk $\{Q \in \mathbb{R}^2 : |P-Q| < r\}$ is contained entirely in D.

P is a *boundary* point of D if and only if, no matter what positive real number r is, the open disk $\{Q \in \mathbb{R}^2 : |P-Q| < r\}$ contains at least one point that is in D and at least one point that is not in D.

Problem 25. *Let D be the rectangle $\{(x,y) \in \mathbb{R}^2 : -2 \le x \le 2$ and $1 \le y \le 3\}$.*

1. *Give an example of an interior point of D.*
2. *Give an example of a boundary point of D.*

Problem 26. *Sketch the region in $\mathbb{R}^2$ given by*

$$\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1, y \le \sqrt{x}\}.$$

The boundary of this region consists of three curves. Write parametric equations for each of these curves.

Problem 27. *Consider the region D in $\mathbb{R}^2$ given by*

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

1. *Sketch D and its boundary.*
2. *What is the dimension of D?*
3. *What is the dimension of the boundary of D?*
4. *Find an implicit equation for the boundary of D.*
5. *Find a parametric equation for the boundary of D.*

Problem 28.
1. *What is the intersection of two planes in $\mathbb{R}^3$? What is its dimension? What is its codimension?*

2. *What is the intersection of a plane and a line in $\mathbb{R}^3$? What is its dimension? What is its codimension?*

Problem 29. *Consider the sets*

$$S = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 6\}$$

and

$$T = \{(x,y,z) \in \mathbb{R}^3 : y-z = 2\}.$$

1. *Sketch S, T, and their intersection. What are their dimensions and codimensions?*
2. *Find an equation for the intersection.*

Problem 30. *Consider the sets*

$$P = \{(x,y,z) \in \mathbb{R}^3 : z = 5\}$$

and

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 4\}.$$

1. *Sketch P, C, and their intersection. What are their dimensions and codimensions?*
2. *Find an equation for the intersection.*

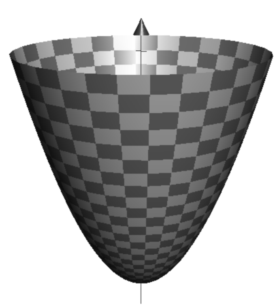
Problem 31. *Sketch the region in $\mathbb{R}^3$ given by*

$$D = \{(x,y,z) \in \mathbb{R}^3 : -1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1\}.$$

Describe the boundary of this region. Write parametric equations for each of the pieces of the boundary.

Problem 32. *Imagine taking a parabola and rotating it around its central axis. It sweeps out a surface called a* ***paraboloid****. (This shape is the traditional shape of car headlights. Nowadays with computer-aided design (CAD) systems, more complicated shapes are sometimes used, but they are based on this shape.)*

1. *What is the dimension of a paraboloid? What is its codimension?*
2. *If we intersect the paraboloid shown with a horizontal plane, what shape will we get?*
3. *If we intersect the paraboloid shown with a vertical plane, what shape will we get?*



Problem 33. *1. Write an implicit equation for a paraboloid.*

2. *Write another implicit equation for a different paraboloid.*

Problem 34. *Let $K : \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by $K(u,v) = (u\cos v, u\sin v, 5-u)$.*

1. *Sketch the range of K for $0 \le u \le 5$ and $0 \le v \le 2\pi$. What is its dimension?*
2. *What is the codimension of the range of K? Write a(n) implicit equation(s) for the range of K.*
3. *On the same graph, sketch and label the curves $K(u, \frac{\pi}{2})$ and $K(3,v)$.*
4. *What is the boundary of the range of K? Write implicit and parametric equations for the boundary.*

Problem 35. *Write a parametric equation for a paraboloid.*

Problem 36. *Suppose that the position of a satellite at time t is given by $s(t) = (t^2, t)$.*

1. *Sketch the path of the satellite from $0 \le t \le 2$.*
2. *On the same graph, sketch three vectors representing the velocity of the satellite at $t = 0$, $t = 1$, and $t = 2$.*
3. *How are these vectors related to the derivative of s?*
4. *Which direction is gravity pulling the satellite?*

Problem 37. *Consider the set $H = \{(x,y,z) \in \mathbb{R}^3 : x^2 - y^2 + z^2 = 1\}$.*

1. *If we plug $y = 1$ into the equation for H, what do we get? What does this tell us about the geometry of H?*
2. *What happens if we plug in other constants for y? What does this mean about the geometry of H?*
3. *Same questions for $x = 0$.*
4. *Sketch H.*

Problem 38. *Consider the set $R = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 - y^2\}$.*

1. *If we plug $y = 1$ into the equation for R, what do we get? What does this tell us about the geometry of R?*
2. *What happens if we plug in other constants for y? What does this mean about the geometry of R?*
3. *Same questions for $x = 0$.*
4. *Which of the graphs below shows R?*

Chapter 3

Optimization

3.1 Derivatives

To describe and predict the world around us mathematical models can be used. Most models don't describe the real world perfectly but they can still be useful, and they are better than nothing.

A company manufactures two items that are sold in two separate markets where it has a monopoly. The quantities q_1 and q_2 demanded by customers, and the prices p_1 and p_2 (in dollars), of each item are related by $p_1 = 700 - q_1$, and $p_2 = 500 - q_2$. Thus, if the price for either item increases, the demand decreases. The companys total production cost is given by $C = 16 + q_1 q_2$.

Problem 39. 1. *Find an equation for the profit function* $P(q_1, q_2)$.

2. *What is a reasonable domain of* $P(q_1, q_2)$*?*

Problem 40. 1. *Create a table that shows values of P for different values of* q_1 *and* q_2.

2. *Using the table, predict for which values of* q_1 *and* q_2 *the profit is largest.*

3. *Using the table, predict for which values of* q_1 *and* q_2 *the profit is smallest.*

Problem 41. 1. *Assume that* $q_1 = 0$ *(what happened to the company??) and draw a graph of* $P(0, q_2)$.

2. *Assume that* $q_2 = 0$ *(what happened now??) and draw a graph of* $P(q_1, 0)$.

3. *Draw a graph of the function* $P(q_1, q_2)$.

Problem 42. *Use your knowledge from calculus 1 to find critical points of* $P(0, q_2)$.

Problem 43. *Recall that in calculus1 the derivative of a function represents the slope of a tangent line. Use that idea to visualize a tangent line to the graph of* $P(q_1, q_2)$ *(using play dough and a toothpick) that shows (and justifies) the critical point you just found. Draw the result carefully or take a picture and describe it.*

Problem 44. *Use your knowledge from calculus 1 to find critical points of* $P(q_1, 0)$. *Visualize a tangent line to the graph of* $P(q_1, q_2)$ *(using play dough) that shows (and justifies) the critical point you just found. Draw the result carefully or take a picture and describe it.*

Problem 45. *Do you think you found a critical point of the function $P(q_1,q_2)$ yet? Why or why not?*

Problem 46. *Choose a different constant for q_1 (not zero this time) and compute the maximum of $P(q_1,q_2)$, as above.*

Problem 47. *Do problem 46 two more times, choosing different constants for q_1.*

Problem 48. *Choose a different constant for q_2 (not zero this time) and compute the maximum of $P(q_1,q_2)$, as above.*

Problem 49. *Do problem 48 two more times, choosing different constants for q_2.*

Problem 50. *Describe what you notice in the above process of finding maxima. Find a process that allows you to know for sure where the maximum of $P(q_1,q_2)$ is (now q_1 and q_2 are* **both** *variables). Now use a geometric argument to expalain why this process will always work.*

Problem 51. *Use partial derivatives to find the points (q_1,q_2) where P has vanishing partial derivatives. We call these points* critical points. *Do you know for sure that this critical point is a maximum? Explain.*

We want to be able to tell if our critical point is really a maximum or not.

Problem 52. *Recall from calculus 1 the different possibilities for critical points of a function $f:\mathbb{R}\to\mathbb{R}$. What could f look like close to a critical point? Draw pictures of the graph of f for the different possibilities.*

Problem 53. *Now take play dough and create all the different ways the graph of a function $f:\mathbb{R}^2\to\mathbb{R}$ can look like around a critical point.*

Problem 54. *The temperature at the point (x,y) in a thunder storm region is given by*

$$T(x,y) = 50 + x^4 + y^4 - 4xy.$$

Find the point(s) (x,y) where the temperature is the lowest. What is the lowest temperature?

3.2 Second Derivatives

Problem 55. *Recall from calculus 1 several strategies that help you determine whether a critical point is a maximum, a minimum or neither. Explain why each strategy works.*

Problem 56. *Use second derivatives to decide whether the critical point of the function* $P(x,y) = x^2 + y^2 - xy$ *is a maximum, a minimum or neither. Explain your strategy in detail. Make sure to check your work.*

Problem 57. *Use second derivatives to decide whether the critical point of the profit function* $P(x,y) = x^2 + y^2 - 4xy$ *is a maximum, a minimum or neither. Make sure to check your work.*

Problem 58. *Compute the mixed partial derivative* P_{xy} *of the function* $P(x,y) = x^2 + y^2 - xy$*. Now make sense of your answer using* P_x *and how it changes in y-direction. Draw one (or several) pictures to explain your thinking.*

Problem 59. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function in problem 56.*

Problem 60. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function in problem 57.*

Problem 61. *Compute the value of* $D = P_{xx}P_{yy} - (P_{xy})^2$ *at the critical point of the function* $f(x,y) = -x^2 - y^2 + 10$.

Problem 62. *Look back at problems 59, 60, and 61. Make a conjecture how you could use D to predict whether a critical point of a function is a maximum, a minimum or a saddle.*

Problem 63. *Compute the critical points of the function*

$$f(x,y) := 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4.$$

Are they maxima, minima or saddle points? Check your work.

Problem 64. *Using a few examples, compare the functions* P_{xy} *and* P_{yx}*. What do you noctice?*

3.3 Tangent Planes

Problem 65. *Recall from calculus 1 what a linear approximation of a function $f : \mathbb{R} \to \mathbb{R}$ around a point p is.*

Problem 66. *Now generalize this idea and explain what a linear approximation of a function $f : \mathbb{R}^2 \to \mathbb{R}$ around a point p is. (Use play dough or draw a picture.)*

One interesting question is if every function has a linear approximation. Let's look back to calculus 1 first.

Problem 67. *Recall from calculus 1 the definitions of a continuous functions, and of a differentiable function. Use the definitions to draw graphs of functions from $\mathbb{R}$ to $\mathbb{R}$ that are*

- *not continuous at some point a*
- *not differentiable at some point a*
- *continuous but not differentiable at some point a. Explain how you can use the tangent line through the point $(a, f(a))$ to graphically show that f is not differentiable at a.*

Problem 68. *Create with play dough the graph of a function from $\mathbb{R}^2$ to $\mathbb{R}$ that is*

- *not continuous at some point (a,b)*
- *not differentiable at some point (a,b)*
- *continuous but not differentiable at some point (a,b). Explain how you can use the tangent plane through the point $(a,b, f(a,b))$ to graphically show that f is not differentiable at (a,b).*

Problem 69. *Draw the graph of the function $f(x,y) = \sqrt{x^2+y^2+1}$. Find two vectors that are tangent to the graph of f at the point $(1,1, f(1,1))$. Hint: You can think in slices and use partial derivatives!*

Problem 70. *Find the equation of the tangent plane at the point $(1,1, f(1,1))$ in parametric representation. Check your work, for instance by drawing both the original graph and the plane that you found (using a computer algebra system).*

Problem 71. *Find the equation of the tangent plane at the point $(1,1, f(1,1))$ in implicit representation. Check your work.*

Problem 72. *Generalize your strategy from problem 69: Given* **any** *function $f(x,y)$ find the parametric and the implicit equation of the tangent plane at the point (a,b).*

3.4 Directional Derivatives

Problem 73. *It is winter in New England. Imagine a bug crawling along the floor of a room that is 7x7 square meters large. Figure 3.1 shows the contour plot of the heat of the floor in Fahrenheit.*

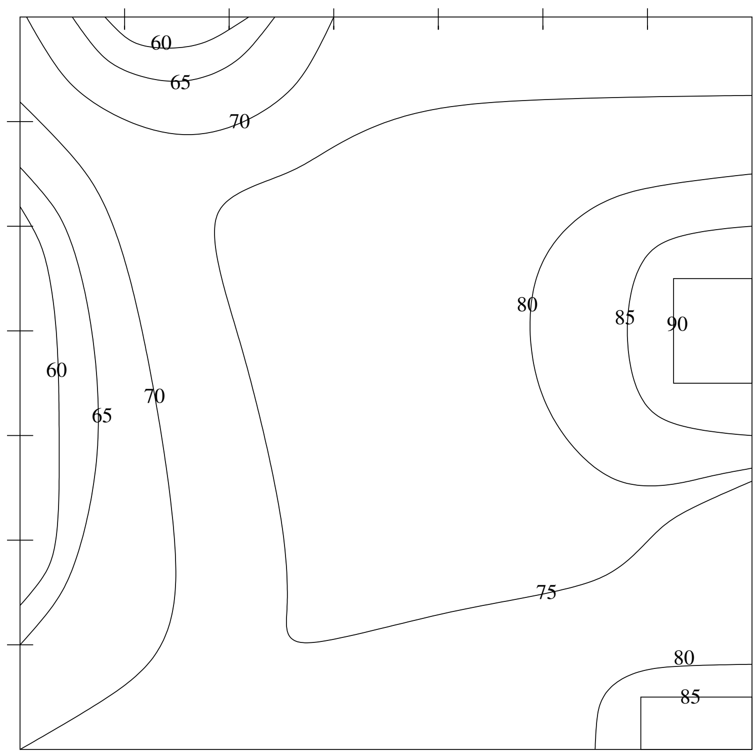


Figure 3.1: Heat Contour

1. *Where is the door in the room? Why?*
2. *Where are the windows? Why?*
3. *Where is the heater? Why?*
4. *What might be standing in the lower right corner of the room?*
5. *The bug would like to crawl from the window to the right wall along a path with the least amount of temperature changes. Draw two possible paths into the contour diagram.*

Problem 74. *Imagine the bug was crawling along the walls. Draw the graphs of the temperature functions ($\mathbb{R} \to \mathbb{R}$) along each wall of the room. Are you able to draw precise graphs? Explain.*

Problem 75. *Figure 3.2 shows the line segment that the bug decided to crawl along. Find the average rate of change of the temperature between points P and Q that the bug experiences. What are the units?*

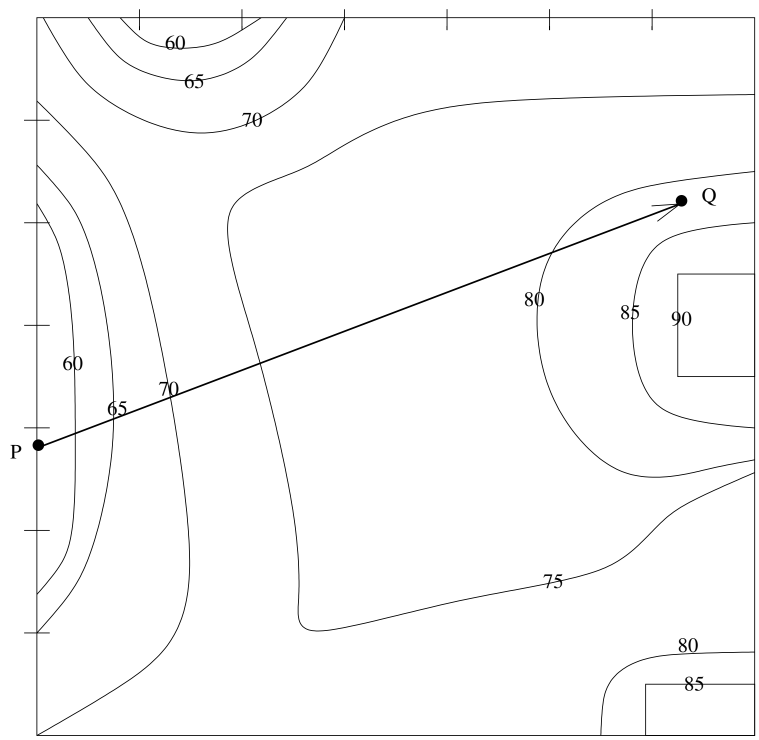


Figure 3.2: Bug on the Floor

Problem 76. *Estimate the instantaneous rate of change of the temperature at point P that the bug experiences when it is preparing to crawl towards Q. How can you improve your estimate? Which information would you need to make your estimate even better?*

Problem 77. *Find a limit expression* $\lim_{h\to 0}\cdots$ *that computes the instantaneous rate of change of the temperature T at point $P=(a,b)$ that the bug experiences (when it faces direction $Q-P=(u_1,u_2)$).*

Problem 78. *How can the directional derivative help the bug find a path with the least amount of temperature changes?*

Problem 79. *To play a bit with the definition, compute the directional derivative of $f(x,y)=4x^2+y$ at the point $(a,b)=(1,2)$ in the direction u for each u defined below.*

1. $u=(0,1)$
2. $u=(1,0)$
3. $u=(1,1)$
4. $u=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$

Problem 80. *The directional derivative is not easy to compute using the definition you just invented (especially by a bug...). We need to find an expression that is easier to compute. Use the tangent plane $f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$ with $x=a+hu_1$, $y=b+hu_2$ to show that*

$$f_u(a,b)\approx\frac{f_x(a,b)u_1+f_y(a,b)u_2}{|(u_1,u_2)|}.$$

Problem 81. *Suppose you (or the bug) are computing a directional derivative. Why is it easier to use the expression in problem 80 than to use the definition?*

Problem 82. *The bug is crawling along the floor of a room from point $(2,\frac{\pi}{4})$ in direction of the vector $v=(1,1)$. The temperature on the floor is given by the function $f(x,y)=60+(x-2)\sin(xy)$ where $0\leq x,y\leq 3$. Find the instantaneous rate of change of temperature that the bug is experiencing. Use a contour diagram to check if your answer is reasonable.*

Problem 83. *To simplify the expression in problem 80 (and make it easier to remember) rewrite it using vectors and a dot product.*

(f_x,f_y) is called the *gradient* of f, denotated by ∇f and we will now explore what other properties the gradient has.

3.5 Gradient

Problem 84. *Explore the relationship between the gradient and tangent vectors of level curves. Draw the contour graph of $f(x,y) = 60 + (x-2)\sin(xy)$ for $0 \leq x, y \leq 3$ using a computer algebra system and draw gradient vectors at several points (we call this a* vectorfield*) into the contour graph. What do you notice? Make a conjecture.*

Problem 85. *It is winter and the bug would like to get to the heater in the room in Figure 3.1 as fast as possible. It entered the room through the window. In which direction does the bug need to go? Draw a possible bug path into the contour graph.*

Problem 86. *Prove your conjecture from problem 84.*

Problem 87. *You want to build a "compass" for the bug in the room. Our compass will always point in the direction of the gradient vector. Answer the following questions to learn how to orient yourself in the room.*

1. *Is the gradient vector 2-dimensional so we can actually build a "normal" compass?*
2. *How can we use the compass to stay just on one level curve? Explain.*
3. *How can we use the compass to walk towards warmer (the warmest?) temperature? Explain.*
4. *How can we use the compass to walk towards colder (the coldest?) temperature? Explain.*

Problem 88. *By changing the direction of u we can make the directional derivative larger and smaller. Can we make it arbitrarily large and small? If not, can you predict a maximum or minimum?*

Problem 89. *Draw the graph (or contour graph) of a temperature function in a square room (with no physical obstructions) in which the following situation can take place: If the bug uses the gradient compass to find the warmest spot in the room, it might never reach it. Explain how this can happen.*

Problem 90.

1. *Let $f(x,y) = 100x^{0.3}y^{0.7}$. Find the direction in which f is increasing most rapidly at the point $(500, 1500)$.*
2. *Suppose your uncle runs a print shop and (somehow) knows that if he has $\$1000x$ invested in labor and $\$1000y$ invested in equipment, his shop can produce $f(x,y) = 100x^{0.3}y^{0.7}$ units of printed materials. He currently has $\$500,000$ invested in labor and $\$1,500,000$ invested in equipment. He wins 1000 in the lottery that he plans to invest in the print shop. He knows you've been studying calculus and comes to you for advice on how to allocate the money. Would you advise him to put more of it into labor, more into equipment, or equal amounts into each? Why?*

3.6 Boundaries, Think Global

Problem 91. *Recall from calculus 1 the definitions of local and global maximum and minimum. Then do the following:*

1. *Draw the graph of a function* $f : D \subset \mathbb{R} \to R \subset \mathbb{R}$ *that has 2 local maxima but no global maximum.*

2. *Draw the graph of a function* $g : D \subset \mathbb{R} \to R \subset \mathbb{R}$ *that has a global maximum at* x *with* $f'(x) \neq 0$.

3. *Draw the graph of a function* $h : D \subset \mathbb{R} \to R \subset \mathbb{R}$ *that has infinitely many global minimums.*

Problem 92. *Let's think about what happens if you have two variables.*

1. *Draw the contour graph of a function that has 2 local maxima, one of them being the global maximum.*

2. *Draw the contour graph of a function that has 2 local maxima, but there is no global maximum.*

3. *Draw the contour graph of a function that has 2 local maxima, but the global maximum is a point on the boundary of the domain.*

4. *Draw the contour graph of a function* f *that has a global maximum at* (x, y) *with* $\nabla f(x, y) \neq 0$.

Problem 93. *The profit of a company is given by* $P(x, y) := -(x-8)^3/3 + 5(x-8)^2 - y^2/2 + 8y$ *for* $0 \leq x \leq 20$, $0 \leq y \leq 15$. *Find the global maximum and minimum of P if they exist. Remember to check what's happening on the boundary of the domain.*

Problem 94. *Let* $f(x, y) = x^2(y+1)3 + y^2$. *Show that f has only one critical point, and that point is a local minimum but not a global minimum. Contrast this with the case of a function with a single local minimum in one-variable calculus.*

3.7 Lagrange Multipliers

Problem 95. *The profit of a company is given by* $P(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$. *For technical reasons the quantities* x *and* y *can only be produced if* $x+y=4$. *Draw the contour graph of* P *and draw the constraint* $x+y=4$ *into the same graph. Use the picture to estimate the global maximum of* P.

Problem 96. *Now try computing the global maximum and explain why this is difficult to do.*

Problem 97. *Draw the gradient of* P *into the contour graph but use only vectors starting at the constraint* $g(x,y) = x+y = 4$. *Then draw the gradient of* g *into the same picture, again using only vectors starting at the constraint line. What do you notice happening at the maximum?*

Problem 98. *Use the above example to create a general procedure to find points that are potential maximums or minimums. Explain why this procedure makes sense.*

Problem 99. *The profit of a company is given by* $P(x,y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$. *For technical reasons the quantities* x *and* y *can only be produced if* $x+y=4$. *Use the procedure you developed above to find the global maximum of* P.

3.8 More Variables

Problem 100. *We want to think about how can we represent and visualize functions. Give examples of functions and several representations for the following cases:*

1. $f : \mathbb{R} \to \mathbb{R}$,
2. $f : \mathbb{R}^2 \to \mathbb{R}$,
3. $f : \mathbb{R}^2 \to \mathbb{R}^2$,
4. $f : \mathbb{R}^3 \to \mathbb{R}$,
5. $f : \mathbb{R}^4 \to \mathbb{R}$.

Problem 101.
1. *Let $f : \mathbb{R}^3 \to \mathbb{R}$. In how many dimensions does the graph of f live? How do you know?*
2. *Let $f : \mathbb{R}^3 \to \mathbb{R}$. In how many dimensions does the contour graph of f live? How do you know?*

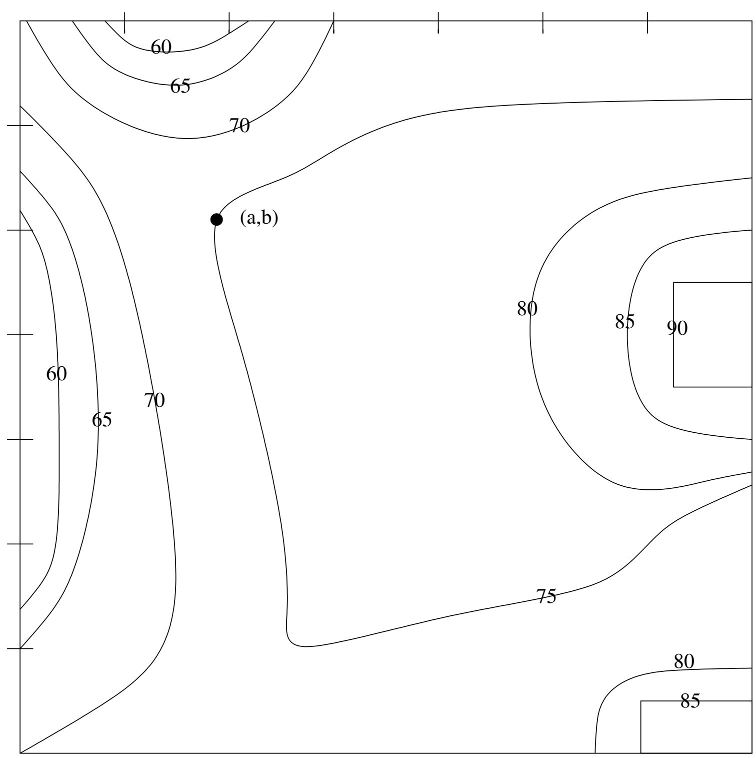


Figure 3.3: Contour Graph of f

Problem 102. *To become more flexible with contour graphs, imagine the bug is standing at point (a,b) in the contour plot of $f : \mathbb{R}^2 \to \mathbb{R}$ in figure 3.3. Draw the contour plot of the tangent plane at $(a,b,f(a,b))$ into the same picture. Use different colors for the different level curves of the tangent plane.*

1. *The bug starts out crawling from point (a,b) in direction $\nabla f(a,b)$. Describe what the bug sees.*

2. *The bug starts out crawling from point (a,b) perpendicular to the direction $\nabla f(a,b)$. Describe what the bug sees.*

Problem 103. *For functions $f(x,y)$ we used tangent planes to approximate the function around a point (a,b). What is the equivalent of a tangent plane, if we have 3 or more input variables?*

Problem 104. *For functions $f(x,y)$ we used level curves in a contour graph to visualize the function. What is the equivalent of a level curve if we have 3 or more input variables?*

Problem 105. *Imagine the bug is flying through the point (a,b,c) in the contour plot of some heat function $f:\mathbb{R}^3 \to \mathbb{R}$. Imagine the contour graph of the tangent space at $(a,b,c,f(a,b,c))$ around the bug. Again, the different level surrfaces of the tangent space have different colours.*

1. *The bug starts out flying from point (a,b,c) in direction $\nabla f(a,b,c)$. Describe what the bug sees.*

2. *The bug starts out flying from point (a,b,c) perpendicular to the direction $\nabla f(a,b,c)$. Describe what the bug sees.*

Problem 106. *Describe how the tangent space is a linear approximation of a function $f:\mathbb{R}^3 \to \mathbb{R}$.*

Problem 107. *Now we have some sense of how to visualize tangent spaces. But sometimes we want to compute the linear approximation. Use partial derivatives to find a (parametric or implicit) equation of the tangent space of a function $f(x,y,z)$ at some point (a,b,c). Hint: look back at the tangent plane equations.*

Problem 108. *Find any critical points of the following function*

$$f(x,y,z) := \frac{5}{6}x^2 + 4x + 16 - \frac{7}{3}xy - 4y - \frac{4}{3}xz + 12z + \frac{5}{6}y^2 - \frac{4}{3}zy + \frac{1}{3}z^2.$$

Can you tell if they are maxima, minima or saddle points? Which of your prior techniques seem to generalize and which don't?

Chapter 4

Integration

Problem 109. *Estimate the average temperature of one wall in the room in figure Heat Contour 1.*

Problem 110. *Recall the definition of the definite integral using Riemann sums, from first year calculus. How does this relate to Problem 109?*

Problem 111. *How could we generalize this idea to estimate the average temperature of the whole floor?*

Problem 112. *Use a technique similar to what you did in Problem 111 to estimate the volume over the rectangle* $\{(x,y) : 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2\}$ *and under the graph of the function* $f(x,y) = x+y+1$.

Problem 113. *How could we generalize this idea to compute the volume under the graph of a general (positive) function* $f(x,y)$ *above a rectangular region?*

Problem 114. *Now imagine that we have a rectangular cracker spread unevenly with peanut butter. Let* $PBD(x,y)$ *represent the density (in grams per square centimeter) of the peanut butter x centimeters from the left edge of the cracker and y centimeters from the bottom of the cracker. Use a technique similar to what you did in Problem 111 to write the total amount of peanut butter (what units?) on the whole cracker in terms of* $PBD(x,y)$.

What you have just developed is the definition of the *double integral* of the function f over the rectangular region R. The notation we use for this is

$$\iint_R f \, dA.$$

Problem 115. *Recall from single variable calculus that we could compute the area of a region using a 'dx' integral or a 'dy' integral.*

1. *Use an integral with respect to x to compute the area of the triangle with vertices* $(0,0)$, $(4,0)$, *and* $(4,3)$.
2. *Compute the same area, this time using an integral with respect to y.*
3. *Check your answer using* $\frac{1}{2}bh$*!*

The way we have set up the integral suggests that the following theorem is true. Actually proving the theorem requires showing that you can move sums and limits around in a way that is subtle and dangerous.

Theorem 1. *If D is the rectangle $\{(x,y) \in \mathbb{R}^2 : a \le x \le b$ and $c \le y \le d\}$, then*

$$\iint_D f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx.$$

Problem 116. *Explain in your own words what Theorem 1 says.*

Problem 117. *Let D be the rectangle $\{(x,y) \in \mathbb{R}^2 : 1 \le x \le 3$ and $2 \le y \le 5\}$. Evaluate*

$$\iint_D x^2 - 3y\, dA.$$

Problem 118. *Evaluate $\int_0^4 \int_0^1 x^2 e^{2y} + y\sqrt{x}\, dy\, dx$ and sketch the region of integration.*

Problem 119. *Evaluate $\int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \cos(x-y)\, dx\, dy$ and sketch the region of integration.*

Problem 120. *Evaluate $\int_0^3 \int_0^2 \int_0^6 x+y+z\, dz\, dy\, dx$ and sketch the region of integration.*

Problem 121. *Evaluate $\int_{-3}^3 \int_{x^2}^9 y+2x\, dy\, dx$ and sketch the region of integration.*

Problem 122.

1. *Evaluate $\int_{-1}^5 \int_0^4 x^2 - xy\, dx\, dy$ and sketch the region of integration.*
2. *Evaluate $\int_0^4 \int_{-1}^5 x^2 - xy\, dy\, dx$ and sketch the region of integration.*
3. *Compare and contrast your answers.*

Problem 123.

1. *Evaluate $\int_0^5 \int_x^5 y^2\, dy\, dx$.*
2. *Evaluate $\int_x^5 \int_0^5 y^2\, dx\, dy$.*
3. *Evaluate $\int_0^5 \int_y^5 y^2\, dx\, dy$.*
4. *Evaluate $\int_0^5 \int_0^y y^2\, dx\, dy$.*
5. *Compare and contrast your answers. Where possible, sketch the region of integration.*

Problem 124. *Evaluate $\int_0^4 \int_0^{\frac{3x}{4}} 1\, dy\, dx$. Compare and contrast with problem 115.*

Problem 125. *Evaluate $\int_0^4 \int_0^{\frac{3x}{4}} y^2\, dy\, dx$. Compare and contrast with problem 124.*

Problem 126. *Evaluate* $\int_0^5 \int_0^4 \int_0^{\frac{3x}{4}} 1\, dy\, dx\, dz$. *Compare and contrast with Problems 115 and 124.*

Problem 127. *Let D be the plane region whose boundary consists of the curves $y = x$ and $y = x^2$. Compute* $\iint_D 1 + xy\, dA$ *two ways (fill in the blanks):*

1. $\int_{--}^{--} \int_{--}^{--} \_\_\_\, dy\, dx$
2. $\int_{--}^{--} \int_{--}^{--} \_\_\_\, dx\, dy.$

Problem 128. *Let R be the plane region bounded by $y = x^2$ and $y = 4$. Find the area of R three ways (fill in the blanks):*

1. *As a single integral,* $\int_{--}^{--} \_\_\_\, dx.$
2. *As a double integral,* $\int_{--}^{--} \int_{--}^{--} \_\_\_\, dy\, dx$
3. *As a double integral,* $\int_{--}^{--} \int_{--}^{--} \_\_\_\, dx\, dy.$

Problem 129. *Evaluate* $\int_{-1}^{1} \int_{-1}^{1} \int_{0}^{x^2+y^2} z\, dz\, dy\, dx$ *and sketch the region of integration.*

Problem 130. *Evaluate* $\int_0^6 \int_{x/3}^{2} x\sqrt{y^3+1}\, dy\, dx.$

Problem 131. *Sketch the region of integration for* $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx.$

Problem 132. *Sketch the region of integration for* $\int_0^2 \int_0^x \int_0^{4-x^2} f(x,y,z)\, dz\, dy\, dx.$

Problem 133. *Consider the tetrahedron T whose boundary consists of the planes $3x + 2y + z = 6$, $x = 0$, $y = 0$, and $z = 0$.*

1. *Sketch T.*
2. *How many ways are there to choose the order of integration for* $\iiint_T f(x,y,z)\, dV$*?*

Problem 134. *Find the limits of integration for all of the orders in Problem 133.*

Problem 135. *Find the mass in grams of the solid cone above $z = \sqrt{x^2+y^2}$ and below $z = 5$, if its density in g/cm^3 is given by $\rho(x,y,z) = x + y + z$. Assume that x, y, and z are measured in cm.*

Polar Coordinates

Like the more familiar rectangular (x-y), the polar coordinate system is a way to specify the location of points in the plane. Some curves have simpler equations in polar coordinates, while others have simpler equations in the rectangular coordinate system.

Definition 3. *Given a point P in the plane we associate P with an ordered pair (r,θ) where r is the distance from $(0,0)$ to P and θ is the angle between the $x-$axis and the ray $\overrightarrow{OP}$ as measured in the counter-clockwise direction.*

Problem 136. *Suppose (x,y) is a point in the plane in rectangular coordinates. Write formulas (in terms of x and y) for*

1. *the distance r from the origin to (x,y) and*
2. *the angles between the positive x-axis and the line containing the origin and P.*

Problem 137. *Plot the following points in the polar coordinate system and find their coordinates in the rectangular coordinate system.*

1. $r = 1, \theta = \pi$
2. $r = 2, \theta = \frac{2\pi}{3}$
3. $r = 3, \theta = \frac{5\pi}{4}$
4. $r = -3, \theta = \frac{\pi}{4}$
5. $r = 2, \theta = -\frac{\pi}{6}$

Problem 138. *Suppose a point in the plane has r and θ as its polar coordinates and (x,y) are its coordinates in the rectangular coordinate system. Write formulas for x and y in terms of r and θ.*

Problem 139. *Sketch the region of the plane where $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.*

Problem 140. *Express this plane region in terms of polar coordinates: $x^2 + y^2 < 1$ and $x \geq 0$.*

Problem 141. *Recall that the area of the sector of a circle with radius r spanning θ radians is $A = \frac{1}{2}r^2\theta$. Let $0 < r_1 < r_2$ and $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$. Sketch the region in the first quadrant bounded by the two circles $r = r_1$, $r = r_2$, and the two lines $\theta = \theta_1$, and $\theta = \theta_2$. Show that the area of the bounded region is $\left(\dfrac{r_1 + r_2}{2}\right)(r_2 - r_1)(\theta_2 - \theta_1)$.*

Theorem 2. *If f is integrable on a region D, then*

$$\int_D f\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)\, r\, dr\, d\theta.$$

Problem 142. *Convert to polar and compute $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \sin(x^2 + y^2)\, dx\, dy$.*

Problem 143. *How is Theorem 2 related to Problem 141?*

Problem 144. *Use polar coordinates to find the volume of the solid under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$.*

Problem 145. *Find the volume under the paraboloid $z = 9 - x^2 - y^2$, and above the $z = 0$ plane in four ways: As a double integral in rectangular coordinates (use your calculator to do the integral), a double integral in polar coordinates, a triple integral in rectangular coordinates (calculator!), and a triple integral in "cylindrical" (polar with z) coordinates.*

Problem 146. *The function $f(x) = e^{-x^2}$ is very important in probability and statistics. It is sometimes called a bell curve, or a Gaussian or normal curve.*

1. *Graph f and explain how this kind of shape might be used to model, for example, how many midshipmen there are of a given height. Give an example of another situation where a curve of this shape would give a distribution of values.*

2. *In these two situations, what would a definite integral $\int_a^b e^{-x^2}\,dx$ represent?*

Problem 147. *Unfortunately, the function $f(x) = e^{-x^2}$ has no antiderivative in terms of functions we know and love! So we cannot use the Fundamental Theorem of Calculus to calculate its integral. Bummer. However...*

1. *Let D_R be the disk $x^2 + y^2 \le R^2$. Evaluate $\int_{D_R} e^{-(x^2+y^2)}\,dA$. Call this number I_R.*

2. *What is $\lim_{R\to\infty} I_R$ and what does it represent?*

3. *Explain why $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.*

4.1 Integrals over Curved Surfaces and Curved Lines

Now we are going to do something that may or may not be tasty: spread peanut butter on a Pringle. The big question is: Suppose we have a curved surface, and something (peanut butter, electric charge, etc.) spread over it unevenly. We would like to set up an integral to represent the total amount of stuff (peanut butter, charge, etc.), based on the density function.

Problem 148. *How, mathematically, might we represent the curved surface? To be very specific, suppose that our surface is the hyperbolic paraboloid (a.k.a. Pringle, a.k.a. saddle surface) from Problem 37. How can we write an equation or equations for this surface?*

Recall that the general idea in integration is that $\int_{object} function\ dSize$ represents the number you get at the end of the following process:

1. Chop the object into pieces.

2. Pick one point on each piece and evaluate the function there to get a number.

3. Multiply this number by the size of its piece. ("Size" here might mean length, or area, or volume. The result of this multiplication should have the units you want to end up with.)

4. Add up the results for all pieces. This gives you an approximation of the final answer.

5. Take the limit as the number of pieces goes to infinity and the size of each piece goes to zero. Unless you are very unfortunate, this limit will exist and give you the number you're looking for.

Problem 149. *Explain how this process corresponds to the definitions we have already seen of* $\int_a^b f(x)\,dx$, $\iint_R f\,dA$ *in rectangular and polar coordinates, and* $\iiint_E f\,dV$.

Problem 150. *Explain how this process corresponds to the Calculus II application of computing volumes using slices.*

Problem 151. *Explain how this process is related to Problem 87, and to computing the amount of peanut butter on a curved surface.*

Problem 152. *What would be a good way to chop a cylinder into pieces? What would the area of each piece be?*

Problem 153. *What would be a good way to chop the plane from Problems 12 and 18? What would the area of each piece be?*

In general if we have a parametric equation $S:\mathbb{R}^2 \to \mathbb{R}^3$ for a surface, an easy way to divide it into pieces is by using the two parameters. Our first set of cuts leaves one variable constant; then we cut across those by leaving the other variable constant.

Computing the area of a piece can be tricky, for two reasons:

1. Different pieces may be different sizes (as in polar coordinates).
2. The cuts might not be perpendicular to each other.

So we use the following ideas to find the area of a small piece:

1. The derivative with respect to a parameter will give a vector tangent to the surface.
2. The cross product of two vectors gives a vector whose magnitude is the area of the parallelogram with the original two vectors as sides.

Putting these ideas together gives us the following.

Theorem 3. *If* $S : [t_1,t_2] \times [u_1,u_2] \to \mathbb{R}^3$ *is a parametric equation for a surface* Q*, and* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *is a function, then*

$$\iint_Q f\,dA = \int_{v_1}^{v_2}\int_{u_1}^{u_2} f(S(u,v)) \left|\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}\right| du\,dv.$$

Problem 154. *Suppose that we have an open-ended metal tube, the shape of a cylinder of radius 3cm and length 5cm, and there is an uneven electric charge on it. We choose coordinates so that the center of the tube is along the z-axis and the bottom of the tube is at* $z = 0$. *In this coordinate system, the electric charge density is* $x+3$ *Coulombs per square centimeter. What is the total electric charge on the tube?*

Problem 155. *Let* P *be the part of the paraboloid with equation* $z = x^2+y^2$ *and* $0 \le z \le 4$*, and let* $f(x,y,z) = x^2+y^2+z^2$. *Evaluate* $\iint_P f\,dA$.

Problem 156. *Suppose that peanut butter is spread on a surface whose shape can be described by the equation* $z = x^2-y^2$ *(see Problem 37), with* $x^2+y^2 \le 1$. *The density of peanut butter, in grams per square centimeter, is given by* $PBD(x,y,z) = 2-z$. *Find the total amount of peanut butter on the surface.*

Now we want to integrate a function over a "curved line," or curve. An example that would be good to think of is of a curved wire, with electric charge unevenly distributed along it. If $f(x,y,z)$ gives the charge density in Coulombs per centimeter, then $\int_C f\ ds$ represents the total charge (in Coulombs) of the whole wire.

Problem 157. *Suppose $\overrightarrow{c} : \mathbb{R} \to \mathbb{R}^3$ is a vector valued function which is differentiable on $[a,\ b]$, and $\overrightarrow{c}(t) = (x(t), y(t), z(t))$. Let s be the arc length of the curve $\overrightarrow{c}(t)$ from the point $(x(a), y(a), z(a))$ to $(x(b), y(b), z(b))$.*

1. *Let $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of $[a,b]$.*
   Show that $s \approx \sum_{i=1}^{n} \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2}$.
2. *Show that $s \approx \sum_{i=1}^{n} (t_i - t_{i-1}) \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}$.*
3. *Explain why this might lead you to believe Theorem 4, below.*

From the previous problem, the following theorem seems plausible. [1]

Theorem 4. *If $\overrightarrow{c} : \mathbb{R} \to \mathbb{R}^3$ is a vector valued function which is differentiable on $[a,\ b]$ so that $\overrightarrow{c}(t) = (x(t), y(t), z(t))$ then the arc length of $\overrightarrow{c}$ on $[a,\ b]$ is $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$.*

If $\overrightarrow{c} : \mathbb{R} \to \mathbb{R}^2$ is a vector valued function which is differentiable on $[a,\ b]$ so that $\overrightarrow{c}(t) = (x(t), y(t))$ then the arc length of $\overrightarrow{c}$ on $[a,\ b]$ is $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$.

Problem 158. *A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius a. If the charge density function is $f(x,y) = kxy$, find the total charge on the wire.*

Problem 159. *Suppose we want to use an integral $\iint_S f\ dA$ to find the area of a surface S. For example, we might need to know how much paint to buy to paint it. What function should we use for f?*

Problem 160. *Check your answer to the previous problem by proving that the surface area of a cylinder is $2\pi rh$.*

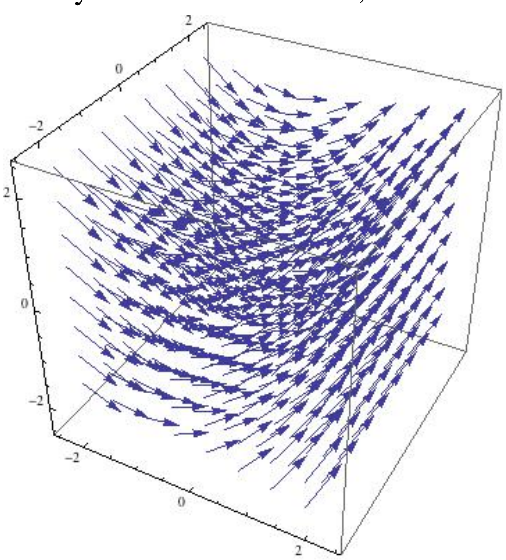
Problem 161. *Suppose we want to use an integral $\int_C f\ ds$ to find the length of a curve C. What function should we use for f? Check your answer by finding the circumference of a circle.*

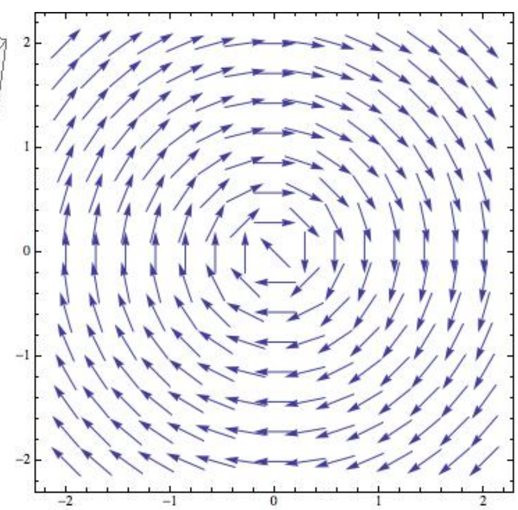
[1] There is actually some subtle magic going on when the derivative limit and the sum limit are taken at the same time! Luckily the Mean Value Theorem you saw in first semester calculus actually tells you these limits can be taken at the same time, provided that $x(t)$ and $y(t)$ are well-behaved functions.

Chapter 5

Vector Fields

Imagine a large body of water, like a river or part of the ocean. The current in different parts of the water may flow faster or slower, or in different directions.





In mathematics, we use a **vector field** to represent the velocity of the water at different points. Vector fields are also used to represent electric and magnetic fields, forces, and velocity of other fluids like air or the dust between stars in a galaxy. Intuitively, a vector field consists of an arrow at each point. We saw an example of a vector field when we computed the gradient in Chapter 3.

Problem 162. *How mathematically, do we represent arrows? Write down an expression for an arrow in the plane that is different at different parts of the plane.*

The general formula for a vector field is $\overrightarrow{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$.

Problem 163. *Sketch the vector field* $\overrightarrow{F}(x,y) = \frac{y}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{-x}{\sqrt{x^2+y^2}}$.

Now suppose we are studying a river. We have a net in part of the river, and want to write an integral to represent the amount of water flowing through the net.

Problem 164. *What units do we want our answer to be in?*

Problem 165. *Suppose the velocity field in the river is* $\overrightarrow{F}(x,y,z) = 3m/s\hat{\imath}$*, and the net can be described by the equation* $S(t,u) = (1,t,u)$ *for* $-2 \le t \le 2$ *and* $0 \le u \le 5$ *(all distances in meters).*

1. *Draw a sketch of the net and the river flow.*

2. *At what rate is water is flowing through the net? Explain your reasoning.*

Problem 166. *Now suppose that the velocity field of the river is the same as in the previous problem, but that the net now has equation $S(t,u) = (t,1,u)$ for $-2 \leq t \leq 2$ and $0 \leq u \leq 5$. At what rate is water flowing through the net? Explain your reasoning.*

Problem 167. *What happens if the net is at an angle to the current?*

Problem 168. *Use properties of the dot product to explain why, if $\overrightarrow{d}$ is a vector and $\hat{u}$ is a unit vector, $\overrightarrow{d} \cdot \hat{u}$ gives the component of $\overrightarrow{d}$ in the $\hat{u}$ direction.*

Definition 4. *The **flux** of $\overrightarrow{F}(x,y,z)$ **across the surface** $S(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$ is defined by*

$$Flux = \iint_S \overrightarrow{F} \cdot \hat{n}\, dA,$$

where $\hat{n}$ is a field of unit vectors perpendicular ("normal") to the surface.

Problem 169. *Remember that cross product we did back in Theorem 3? Which direction does it point? Is it a unit vector?*

Problem 170. *Use the result from the previous problem to explain why*

$$\iint_S \overrightarrow{F} \cdot \hat{n}\, dA = \iint_S \overrightarrow{F} \cdot (\overrightarrow{S}_u \times \overrightarrow{S}_v)\, du\, dv.$$

Problem 171. *Let $\overrightarrow{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by $\overrightarrow{F}(x,y,z) = (1,1,-1)$.*

1. *Compute the flux of $\overrightarrow{F}$ across the square $S(x,y) = (x,y,2)$ for $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.*

2. *Was your answer to the previous problem positive or negative? Is this what you should have expected?*

Problem 172. *Let $\overrightarrow{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by $\overrightarrow{F}(x,y,z) = (0,0,-z)$.*

1. *Find the flux of $\overrightarrow{F}$ down through the surface given by $S(u,v) = (u,v,u^2+v^2)$ where $u^2+v^2 \leq 4$.*

2. *Find the flux of $\overrightarrow{F}$ up through the surface given by $S(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$ where $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.*

3. *Compare and contrast.*

Problem 173. *Compute the flux of the vector field $\overrightarrow{F}(x,y,z) = (x,y,z)$ out of the sphere $x^2+y^2+z^2 = 9$.*

5.1 Divergence and Gauss's Theorem

Definition 5. *The* ***divergence*** *of a vector field* $\overrightarrow{F}(x,y,z) = P(x,y,z)\ \hat{\imath} + Q(x,y,z)\ \hat{\jmath} + R(x,y,z)\ \hat{k}$ *is*

$$\textbf{div}\ \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Problem 174. *Compute the divergence of the vector field* $\overrightarrow{F}(x,y,z) = (x^2 + e^z)\hat{\imath} + (x^3 + e^y)\hat{\jmath} + (\sqrt{x^3 + 5y} - z)\hat{k}$.

Theorem 5 (Gauss's Divergence Theorem)**.** *If* $\overrightarrow{F}$ *is a vector field on the region surrounded by a closed surface, then the flux of* $\overrightarrow{F}$ *out through the surface is equal to the integral of the divergence of* $\overrightarrow{F}$ *over the enclosed region.*

Problem 175. *Write Gauss's Divergence Theorem as an equation with an integral on each side.*

Problem 176. *Redo Problem 173 using Gauss's Divergence Theorem.*

Problem 177. *Wow, that definition of divergence looks kind of like a dot product to me! If it were the dot product of something with* $\overrightarrow{F}$*, what would the something be?*

The Divergence Theorem is an example of a higher-dimensional version of the Fundamental Theorem of Calculus. Both have the general form

$$\int_{\text{object}} \text{derivative of } f = \int_{\text{boundary of object}} f. \tag{5.1}$$

Problem 178. *Explain how each piece of Gauss's Divergence Theorem fits into the framework of Equation 5.1.*

Problem 179. *Write down the equation in the part of Fundamental Theorem of Calculus we use all of the time to evaluate integrals. Explain how each piece of this equation fits into the framework of Equation 5.1.*

5.2 Work and the Fundamental Theorem of Line Integrals.

You will recall from physics that "work = force $\times$ distance." If the force being applied is not in line with the motion of the object, then only the component of the force in the direction of motion contributes to the work– in this case, we have "work = force $\times$ distance $\times \cos\theta$, or "work is the dot product of the force and the displacement."

You may also recall from Calculus II analyzing the amount of work done in situations like stretching a spring, or pumping water out of a tank, where either different parts of the object are moving different distances, or the amount of force varies as you go, or both. In this case, we had something more like "work = $\int_{\text{pieces}}$ (force on a piece) $\times$ (distance for that piece)."

Putting these together lets us compute work in situations where the force is both changing and not in line with the motion, for example when launching a satellite.

Problem 180.

1. *Make a fairly large and careful sketch of the curve* $C(t) = (3\cos t, 3\sin t)$ *for* $0 \le t \le \pi$.

2. *Let* $\overrightarrow{F}(x,y) = -\frac{x^2}{9}\hat{\imath}$. *For each of* $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, *sketch the two vectors* $\overrightarrow{C}'(t)$ *and* $\overrightarrow{F}(C(t))$, *both based at the point* $C(t)$.

3. *For each of the t values in part 2, compute the component of* $\overrightarrow{F}$ *in the* $\overrightarrow{C}'(t)$ *direction and label that point on the graph with this value. What do you notice?*

Definition 6. *The* ***line integral of the vector field*** $\overrightarrow{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ *over the curve C given by equation* $r(t) = (x(t), y(t), z(t))$ *is defined by*

$$\int_C \overrightarrow{F} \cdot d\mathbf{r} = \int_a^b \overrightarrow{F}(r(t)) \cdot r'(t)\, dt$$

Problem 181. *Compute the line integral* $\int_C \overrightarrow{F} \cdot d\mathbf{r}$ *where the curve C is given by* $r(t) = (t^2, t^3, t^2)$ *and* $\overrightarrow{F}$ *is given by* $\overrightarrow{F}(x,y,z) = (x+y, y-z, z^2)$.

Problem 182. *Compute the line integral* $\int_C \overrightarrow{F} \cdot d\mathbf{r}$ *where* $\overrightarrow{F}$ *is given by* $\overrightarrow{F}(x,y) = (1, 2y)$ *and C is the line segment from* $(0,0)$ *to* $(5,5)$.

Problem 183. *Remember* ∇ *from Problem 177? What would the cross product* $\nabla \times \overrightarrow{F}$ *be? This is called the* ***curl*** *of* $\overrightarrow{F}$.

Theorem 6. ***The Fundamental Theorem for Line Integrals*** *Suppose that C is a curve starting at point P and ending at point Q. If the vector field* $\overrightarrow{F}$ *is the gradient of a function g, and everything is continuous, then*

$$\int_C \overrightarrow{F} \cdot d\mathbf{r} = g(Q) - g(P).$$

Problem 184. *Compute the line integral from Problem 183 using Theorem 6.*

If we want to use Theorem 6 to compute a line integral $\int_C \overrightarrow{F} \cdot d\mathbf{r}$, we need to find an "anti-derivative" g of $\overrightarrow{F}$: a function whose gradient is $\overrightarrow{F}$. Not every vector field is the gradient of a function.

Definition 7. *A vector field is said to be* ***conservative*** *if it is the gradient of some function. If a vector field* $\overrightarrow{F}$ *is the gradient of a function g, g is called its* ***potential function***.

Problem 185. *Is* $\overrightarrow{F}(x,y) = (x^2+y^2, 2xy)$ *conservative? If so, find its potential function.*

Problem 186. *Is* $\overrightarrow{F}(x,y) = (xy, x-y)$ *conservative? If so, find its potential function.*

Problem 187. *Prove that if a vector field* $\overrightarrow{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ *is conservative, then* **curl** $\overrightarrow{F} = 0$.

Theorem 7. ***Stokes' Theorem*** *If S is a surface and C is the boundary curve of that surface, and* $\overrightarrow{F}$ *is a vector field which is differentiable on S, then*

$$\int_C \overrightarrow{F} \cdot d\mathbf{r} = \int_S (\mathbf{curl}\,\overrightarrow{F}) \cdot \hat{n}\, dA.$$