DISCOVERING THE ART OF MATHEMATICS

PATTERNS

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

Julian F. Fleron and Philip K. Hotchkiss,
with Volker Ecke and Christine von Renesse

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Preface: Notes to the Explorer

Yes, that's you - you're the explorer.

"Explorer?"

Yes, explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a traditional book. Indeed, this book cannot be read in the traditional sense. For this book is really a guide. It is a map. It is a route of trail markers along a path through part of the world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore this path - to take a surprising, exciting, and beautiful journey along a meandering path through a mathematical continent named the infinite. And this is a vast continent, not just one fixed, singular locale.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by dozens of exercises that closely mimic illustrative examples. Rather, after a brief introduction to the chapter, the majority of each chapter is made up of Investigations. These investigations are interwoven with brief surveys, narratives, or introductions for context. But the Investigations form the heart of this book, your journey. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover the mathematics that is behind music and dance. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking, mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is discovered each day than in any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! *Fermat's Last Theorem*, which is considered in detail in Discovering that Art of Mathematics - Number Theory, was solved in 1993 after 350 years of intense struggle. The 1$ Million Poincaŕe conjecture, unanswered for over 100 years, was solved by **Grigori Perleman** (Russian mathematician; 1966 -). In the time period between when these words were written and when you read them it is quite likely that important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning mathematical skills, mathematical reasoning, and mathematical applications. Arithmetical and statistical skills are useful skills everybody should possess. Who could argue with learning to reason? And we are all aware, to some degree or another, how mathematics shapes our technological society. But there is something more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is one of its driving forces. As the famous **Henri Poincaŕe** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as both a liberal art and as a science. As a powerful science, mathematics shapes our technological society and serves as an indispensable tool and language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in many other fine, accessible books (e.g. [COM] and [TaAr]). Instead, our purpose here is to journey down a path that values mathematics from its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was rescued and preserved in monasteries. Knowledge was categorized into the classical liberal arts and mathematics made up several of the seven categories.[1] During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts - except in the contemporary classrooms and textbooks where the focus of attention has shifted solely to the training of qualified mathematical scientists. If you are a student of the liberal arts or if you simply want to study mathematics for its own sake, you should feel more at home on this exploration than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more. In your exploration here you will see that mathematics is a human endeavor with its own rich history of human struggle and accomplishment. You will see many of the other arts in non-trivial roles: dance and music to name two. There is also a fair share of philosophy and history. Students in the humanities and social sciences, you should feel at home here too.

Mathematics is broad, dynamic, and connected to every area of study in one way or another. There are places in mathematics for those in all areas of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

It is my hope that your discoveries and explorations along this path through the infinite will help you glimpse some of this beauty. And I hope they will help you appreciate Russell's claim that:

> ... The true spirit of delight, the exaltation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Finally, it is my hope that these discoveries and explorations enable you to make mathematics a real part of your lifelong educational journey. For, in Russell's words once again:

> ... What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have served as beacons to people who have explored the continents of mathematics throughout history.

[1] These were divided into two components: the *quadrivium* (arithmetic, music, geometry, and astronomy) and the *trivium* (grammar, logic, and rhetoric); which were united into all of knowledge by philosophy.

Navigating This Book

Before you begin, it will be helpful for us to briefly describe the set-up and conventions that are used throughout this book.

As noted in the Preface, the fundamental part of this book is the Investigations. They are the sequence of problems that will help guide you on your active exploration of mathematics. In each chapter the investigations are numbered sequentially. You may work on these investigation cooperatively in groups, they may often be part of homework, selected investigations may be solved by your teacher for the purposes of illustration, or any of these and other combinations depending on how your teacher decides to structure your learning experiences.

If you are stuck on an investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us: “If thee is no struggle, there is no progress.” Keep thinking about it, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Investigation numbers are bolded to help you identify the relationship between them.

Independent investigations are so-called to point out that the task is more significant than the typical investigations. They may require more involved mathematical investigation, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The Connections sections are meant to provide illustrations of the important connections between mathematics and other fields - especially the liberal arts. Whether you complete a few of the connections of your choice, all of the connections in each section, or are asked to find your own connections is up to your teacher. But we hope that these connections will help you see how rich mathematics’ connections are to the liberal arts, the fine arts, culture, and the human experience.

Further investigations, when included are meant to continue the investigations of the area in question to a higher level. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory.

Within each book in this series the chapters are chosen sequentially so there is a dominant theme and direction to the book. However, it is often the case that chapters can be used independently of one another - both within a given book and among books in the series. So you may find your teacher choosing chapters from a number of different books - and even including “chapters” of their own that they have created to craft a coherent course for you. More information on chapter dependence within single books is available online.

Certain conventions are quite important to note. Because of the central role of proof in mathematics, definitions are essential. But different contexts suggest different degrees of formality. In our text we use the following conventions regarding definitions:

- An *undefined term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An ***informal definition*** is italicized and bold faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.
- A **formal definition** is bolded the first time it is used. This is a formal definition that suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a biographical name appears it is bolded and basic biographical information is included parenthetically to provide some historical, cultural, and human connections.

CHAPTER 1

Introduction - Mathematics as the Art and Science of Patterns

In fact, the answer to the question "What is mathematics?" has changed several times during the course of history... It was only in the last twenty years or so that a definition of mathematics emerged on which most mathematicians agree: mathematics is the science of patterns.

Keith Devlin (; -
)

Virtually all young children like mathematics. They do mathematics naturally, discovering patterns and making conjectures based on observation. Natural curiosity is a powerful teacher, especially for mathematics. Unfortunately, as children become socialized by school and society, they begin to view mathematics as a rigid system of externally dictated rules governed by standards of accuracy, speed, and memory. Their view of mathematics shifts gradually from enthusiasm to apprehension, from confidence to fear. Eventually, most students leave mathematics under duress, convinced that only geniuses can learn it.

National Research Council (; -
)

We encounter patterns all the time, every day: in the spoken and written word, in musical forms and video images, in ornamental design and natural geometry, in traffic patterns, and in objects we build. Our ability to recognize, interpret, and create patterns is the key to dealing with the world around us.

Margorie Senechal (; -
)

What is mathematics? Ask this question of person chosen at random, and you are likely to receive the answer "Mathematics is the study of number." With a bit of prodding as to what kind of study they mean, you may be able to induce them to come up with the description "the science of numbers." But that is about as far as you will get. And with that you will have obtained a description of mathematics that ceased to be accurate some two and a half thousand years ago!

Keith Devlin (; -
)

As the science of abstract patterns, there is scarcely any aspect of our lives that is not affected, to a greater or lesser extent, by mathematics; for abstract patterns are the very essence of thought, of communication, of computation, of society, and of life itself.

Keith Devlin (; -
)

A surprising proportion of mathematicians are accomplished musicians? Is it because music and mathematics share patterns that are beautiful?

Martin Gardner (; -
)

A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

G. H. Hardy (; -

)

Born of man's primitive urge to seek order in his world, mathematics is an ever-evolving language for the study of structure and pattern. Grounded in and renewed by physical reality, mathematics rises through sheer intellectual curiosity to levels of abstraction and generality where unexpected, beautiful, and often extremely useful connections and patterns emerge. Mathematics is the natural home of both abstract thought and the laws of nature. It is at once pure logic and creative art.

Lawrence University Catalogue (; -
)

In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels he ought to know why it appears.

W. W. Sawyer (; -
)

Mathematics, in the common lay view, is a static discipline based on formulas...But outside the public view, mathematics continues to grow at a rapid rate...the guide to this growth is not calculation and formulas, but an open ended search for pattern.

Lynn Arthur Steen (; -
)

What humans do with the language of mathematics is to describe patterns... To grow mathematically children must be exposed to a rich variety of patterns appropriate to their own lives through which they can see variety, regularity, and interconnections.

Lynn Arthur Steen (; -
)

Mathematics can be characterized as the science of patterns... Patterns should be part of every mathematics course at all grade levels.

Patterns: Addenda Series K - 6 of the Curriculum and Evaluation Standards, National Council of Teachers of Mathematics, 1993, p. 1. (; -
)

Should use examples that are elsewhere in the books in this series. At least largely so. Give hooks to other things.

Rainbows - Include these here along with the Descartes quote: I believe that I can now give an account of [atmospheric phenomena], and I have decided to write a small treatise that will include an explanation of the cause of the rainbow, the matter that has given me the greatest difficulty. Rene Descartes, quoted in The Rainbow Bridge, p. 182. There is a hook to this in the problem solving chapter that is upcoming. Nice connections.

Many of the other light patterns can go in here too if need be, I think that there are really not enough activities to have a whole chapter for them. So why not include them.

0.1. What is a Pattern? No real definition. Can even look at Grunbaum and Shephard's opus.

Can we classify different types of patterns?

OK, but now *everything* is a pattern, isn't it?

In the wonderful book Curious Incident of the Dog in the Night-Time the hero, Christopher Boone, a 15 year-old "a mathematician with some behavioral difficulties." His description of the constellation Orion, shown visually in his two versions in Figure 1, is:

> People say that Orion is called Orion beccause Orion was a hunter and the constellation looks like a hunter with a club and a bow and arrow... But this is really silly because it is just stars, and you could join up the dots in any way you wanted, and you could make it look like a lady with an umbrella who is waving, or the coffeemaker which Mrs. Shears has, which is from Italy, with a handle and steam coming out, or like a

dinosaur... And there aren't any lines in space, so you could join bits of Orion to bits of Lepus or Taurus or Gemini and say there were a constellation called the Bunch of Grapes or Jesus or Bicycle (except that they didn't have bicycles in Roman and Greek times, which was when they called Orion Orion). And anyway, Orion is not a hunter of coffeemakers of a dinosaur. It is just Betelgeuse and Bellatrix and Alnilam and Rigel and 17 other stars I don't know the names of. And they are nuclear explosions billions of miles away. And that is the truth.

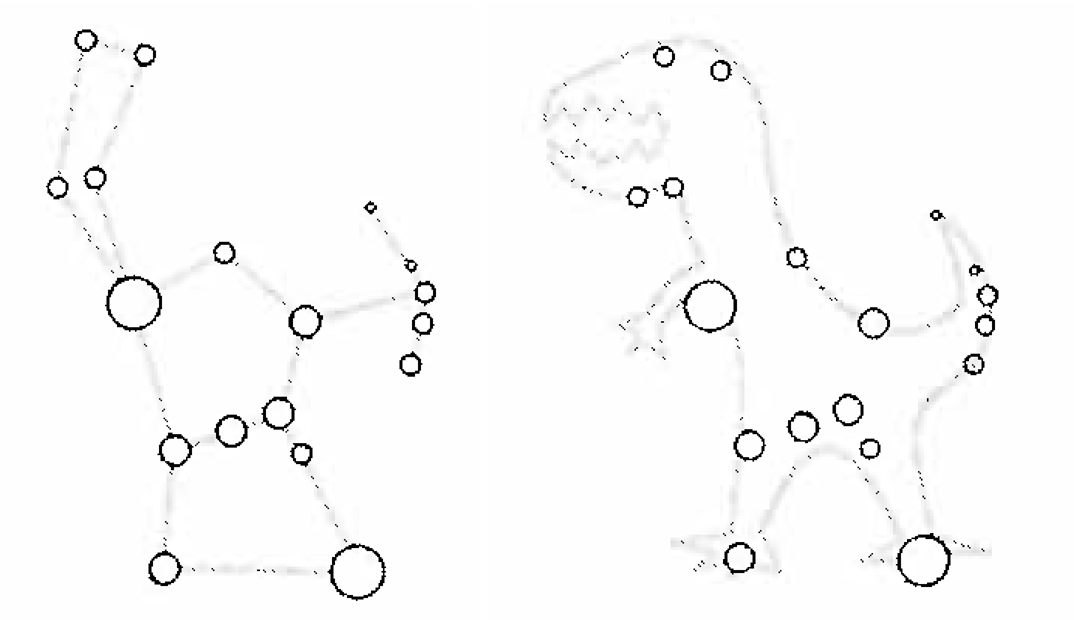


Figure 1. Two different views of the constellation Orion.

Orion is not a pattern.

1. Animal Coat Patterns

I've got spots, I've got stripes too.[1]

Ani DiFranco (American Poet and Folksinger; 1970 -)

Figure 2 shows the coats and surface textures of several animals. A fundamental breakthrough in the understanding of these types of pattern formation was made in 1952 by **Alan Turing** (English mathematician; - 1954).

Turing's work on patterns was essential in a number of areas. Turing is widely considered to be the father of computer science and artificial intelligence. His *Turing test* remains a fundamental tool to evaluate whether a computer can "think". Turing's work with others at Bletchley Park allowed the Allies to break the German secret encryption codes generated by *enigma machines*. These contributions played an essential role in turing the tide of the war in the Atlantic in the favor of the Allies. While highly decorated for his work after the war, Turing was later discovered to be homosexual. He was forced to undergo chemical "treatment", was put under house arrest and eventually killed himself by eating an apple laced with cyanide. **Peter Hilton** (; -) tells us,

[1]From "In or out" on the album "Imperfectly."

FIGURE 2. A cheetah, a leopard, brain coral and hawksbill turtle.

> I.J. Good, a wartime colleague and friend, has aptly remarked that it is forunate that the authorities did not know during the war that [Alan] Turing was a homosexual; otherwise, the Allies might have lost the war.[2]

Returning to Turing's work in developmental biology, Turing's breakthrough was published as "The Chemical Basis of Morphogenisis,"[3] which used *reaction-diffusion equations* to model the pattern formation. Like his algorithms for machines to learn to play chess, computers appropriate to test these equations did not exist during Turing's life. However, as with chess-playing computers, we can now easily experiment with these models.

[2]"Cryptanalysis in World War II – and Mathematics Education," *Mathematics Teacher*, Oct. 1984.

[3]Philosophical Transactions of the Royal Society of London 237 (641): 3772; available online at `http://www.dna.caltech.edu/courses/cs191/paperscs191/turing.pdf`

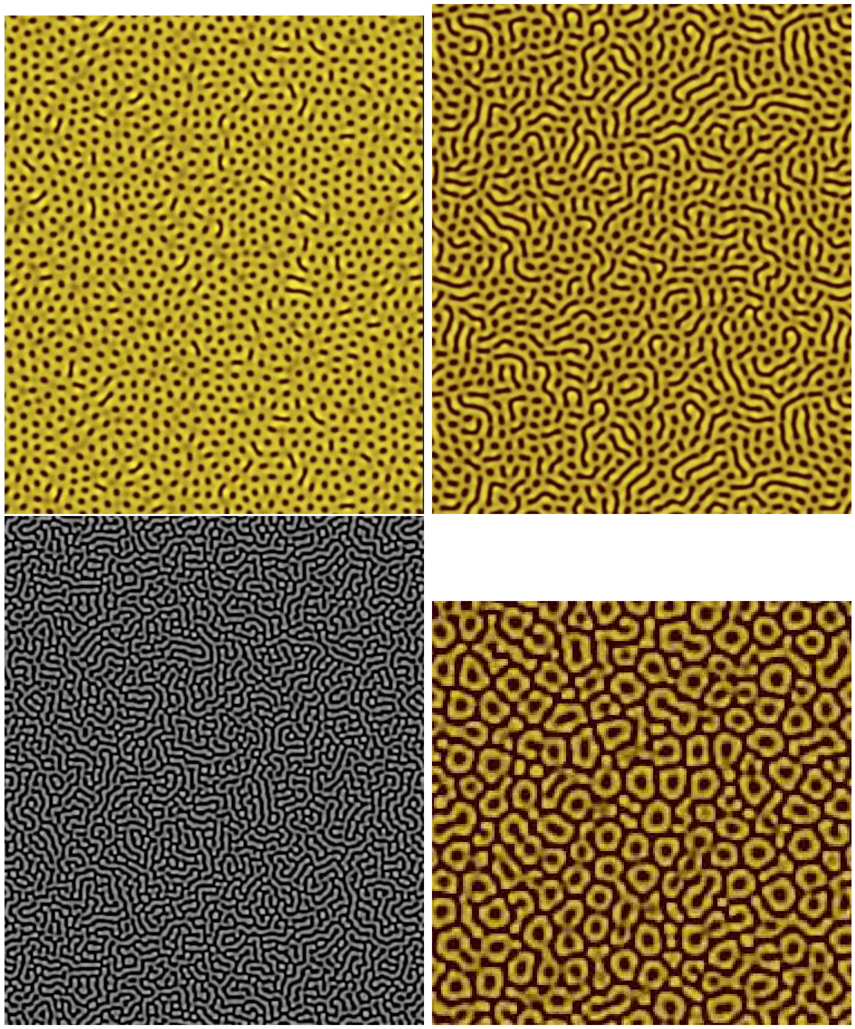


FIGURE 3. Turing reaction diffusion models generated from the applet available at `http://cgjennings.ca/toybox/turingmorph/`; Diffusion constants 3.5 & 16; 3.0 & 22; 1.0 & 22; ?????.

We will explore a simple, *two parameter reaction-diffusion model* using the applet at

`http://cgjennings.ca/toybox/turingmorph/`.

The success of these simple models can be seen in Figure 3 where the coat and surface patterns of the animals pictured in Figure 2 have been modeled.

Begin experimenting with this model by choosing different values for the diffusion constants, the presets and the number of ***iterations*** (steps the model takes). You may record any images you create electronically.[4]

1. Find several animals (different from those pictured above) whose coats or suface patterns you find interesting and you would like to try to model using diffusion-reaction models. (Note: You are

[4]Use a computer screen capture or snapshot; "Print Screen" on a PC and Command-Control-Shift-3 on a Mac.

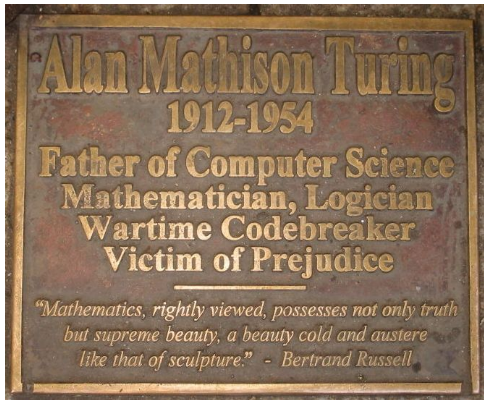


FIGURE 4. Plaque at the Alan Turing Memorial in Manchester, England.

FIGURE 5. Statues of Alan Turing at Bletchley Park and at the Alan Turing Memorial in Manchester, England.

not trying to model the geometric shape of structural surface features like the lattice of scales on a snake, these are formed differently.)

2. For one of your animals, create their coat or surface pattern using the applet. Your final image should be quite close to the original in overall structure. While color is interesting, it is the structure of the pattern that is essential here. (Note: Your pattern should also be from a stable region of the model. If your image is rapidly changing as the model iterates this is not appropriate - animal's coat patterns do not change like this, chameleons notwithstanding.) Capture your image, the settings you used to make it, and identify clearly what animal coat or surface pattern it models.

3. Repeat Investigation **2** for a different animal.

4. Repeat Investigation **2** for a third different animal.

5. Are there some animals or types of coat/surface patterns that you had trouble making using this applet? If so, describe them.

6. Make a "multi-stage pattern", as described on the applet's site, to make a coat/surface pattern that is fundamentally different than others you have already made and which models some animal's coat/surface pattern.

Transits of Venus Get JF pictures. People looking at it.

Kepler's Laws Kepler's Model of the Planets

Discovery of neptune. What is the other planet that plays a role in all of those stories, including Music of the Spheres?

FIGURE 6. Fruit of the Kousa tree; Westfield, MA.

FIGURE 7. Hot sauce pattern.

MUST get flower pictures in here for Fibonacci. How to do that without going into a whole thing that overshadows the number theory book?

FIGURE 8. Waves and Sand in St. Croix. Video available online at ???.

FIGURE 9. School of ??fish; St. Croix.

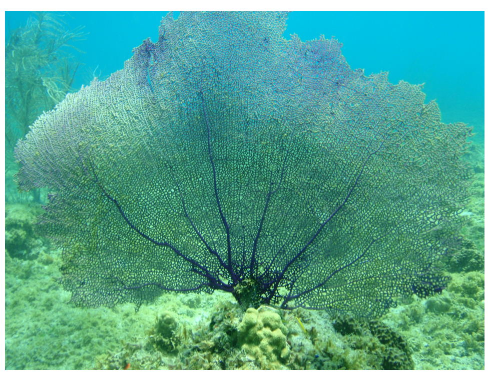


FIGURE 10. Sea coral; St. Croix.

1.1. Transits of Venus. Consider the list of dates below.

- 7 December, 1631
- 4 December, 1639
- 6 June, 1761

FIGURE 11. Cactus flower; St. Croix.

FIGURE 12. Anole; St. Croix.

- 3 June, 1769
- 9 December, 1874
- 6 December, 1882
- 8 June, 2004
- 6 June, 2012

7. Can you see a pattern, or at least an approximate pattern? Explain.
8. If you can see a pattern, predict what the next several dates in this pattern might be.
9. What is a ecclipse?
10. Are there different types of ecclipses? Explain.

FIGURE 13. Palm tree; St. Croix.

11. What are the geometric factors that give rise to ecclipses?
12. For centuries astronomers have been able to predict future ecclipses with great accuracy. While the specific details are quite complex, explain generally how it might be that such predictions can be made.

The dates given above are the dates for the ***transits of Venus***. These are astronomical events akin to an eclipse, only here the intervening body is significantly smaller that the body it "transits" in front of.

13. When looking at the sun from the Earth, what planets can transit in front of the sun?
14. If we were on Jupiter, what planets can transit in front of the sun?
15. From Earth, do you think that transits of Mercury or transits of Venus happen more frequently? Explain why.

NO transits of Venus occurred in the twentieth century. There were exactly two in the twenty-first century. They are pictured below. At the time of writing of this book, 2013, most people who have not seen one of the transits of Venus that just occurred will not live to have the opportunity to see one.

Future dates of transits of Venus are as follow:

- 11 December, 2117
- 8 December, 2125
- 11 June 2247
- 9 June, 2255
- 13 December, 2360
- 10 December, 2368
- 12 June, 2490
- 10 June, 2498
- 16 December, 2603

FIGURE 14. 2004 transit of venus taken at sun-up on the Catawba River near Connelly's Springs, NC by David Cortner (left) and 2012 transit of venus taken at dusk near the town green in Westfield, MA by Julian Fleron.

- 13 December, 2611
- 15 June, 2733
- 13 June, 2741
- 16 December, 2846
- 14 December, 2854
- 16 June, 2976
- 14 June, 2984
- 18 December, 3089

16. Can you think of a reason why the dates of consecutive transits of Venus separated by less than a decade differ by two days and other times they differ by three? Explain.

17. After that last date on the list above, what do you think the next date will be?

In fact, there is only a single transit of Venus in the thirty-first century. And there is only one in the thirty-fourth century as well. ???How to explain this with the planes, Chinese Remainder Theorem type thing? How often are Earth, Venus and Sun all coplanar? Is this about 8 years? And 112 is a multiple of 8. Cool.

Now talk about the importance of transits of Venus historically.

CHAPTER 2

Patterns of the Day

One way to build understanding of the ubiquity of patterns is to begin each class with a "Pattern of the Day." In this chapter we provide a list of patterns that we have found approrpriate for this purpose.

In each there is an indicated pattern and the task is to:

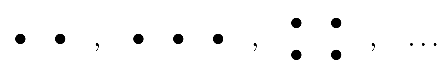
- Guess the next few stages in the pattern.
- Understand why the pattern appears and/or continues.
- Find some context, application, or motivation for the pattern.

Notice how closely these tasks follow the maxim of Sawyer:

> In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels he ought to know why it appears.
>
> **W.W. Sawyer** (; -)

Occasionally there are investigations that follow the pattern. These involve specific prompts that are useful beyond the typical tasks just described.

1. A Prime Pattern



1. Now find a totally different way to continue this pattern.
2. And find another different way to continue this pattern.
3. Exchange your patterns with a peer. Can you understand the patterns the other has found?
4. What is the correct answer?

2. Are you positive?

$$\begin{aligned} 5 \times 3 &= 15 \\ 5 \times 2 &= 10 \\ 5 \times 1 &= 5 \\ 5 \times 0 &= 0 \\ \vdots &= \vdots \end{aligned}$$

5. Does this help you to see why the product of a positive and a negative is a negative? Explain.
6. With some rationale for why the product of a positive and a negative is a negative, can you find a pattern which uses this fact to show why the product of two negatives is a positive? Either do so or explain why you cannot.

3. PatternProducts

$$\begin{gathered}
1 \times 1 = 11 \\
11 \times 11 = 121 \\
111 \times 111 = 12321 \\
1111 \times 1111 = 1234321 \\
11111 \times 11111 = 123454321 \\
111111 \times 111111 = 12345654321 \\
1111111 \times 1111111 = 1234567654321 \\
11111111 \times 11111111 = 123456787654321 \\
111111111 \times 111111111 = 12345678987654321
\end{gathered}$$

$$\begin{gathered}
1 \times 1 + 8 = 9 \\
12 \times 2 + 8 = 98 \\
123 \times 3 + 8 = 987 \\
1234 \times 4 + 8 = 9876 \\
12345 \times 5 + 8 = 98765 \\
123456 \times 6 + 8 = 987654 \\
1234567 \times 7 + 8 = 9876543 \\
12345678 \times 8 + 8 = 98765432 \\
123456789 \times 9 + 8 = 987654321
\end{gathered}$$

$$\begin{gathered}
1 \times 9 + 2 = 11 \\
12 \times 9 + 3 = 111 \\
123 \times 9 + 4 = 1111 \\
1234 \times 9 + 5 = 11111 \\
12345 \times 9 + 6 = 111111 \\
123456 \times 9 + 7 = 1111111 \\
1234567 \times 9 + 8 = 11111111 \\
12345678 \times 9 + 9 = 111111111 \\
123456789 \times 9 + 10 = 111111111
\end{gathered}$$

$$\begin{aligned}
9 \times 9 + 7 &= 88 \\
98 \times 9 + 6 &= 88 \\
987 \times 9 + 5 &= 88 \\
9876 \times 9 + 4 &= 88 \\
98765 \times 9 + 3 &= 88 \\
987654 \times 9 + 2 &= 88 \\
9876543 \times 9 + 1 &= 88 \\
98765432 \times 9 + 0 &= 88
\end{aligned}$$

4. Arithmetical Savants

TFSSTWTFSMTWFSSMWTFS ...

There are great stories of savants (??ok word) with tremendous arithmetical abilities and tremendous memories.

Twins brief story.

Born on a Blue Day and pi and Swedish language and being able to describe how he does arithmetic in terms of art.

We still have very little idea how these people are so brilliant in certain ways and so limited in others.

One amazing thing that savants can do is tell you the day of the week...

7. Why do the days of the week follow the pattern that they do from one year to another?? (Reword this?? How to hide this?)

5. Continued Fractions

$$1+\frac{1}{1}=?$$
$$1+\frac{1}{1+\frac{1}{1}}=?$$
$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}=?$$
$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}=?$$
$$\vdots=\vdots$$

For much more see "The Golden Ratio" from <u>Discovering the Art of Mathematics - Number Theory</u>.

It is remarkable to note that

$$\sqrt{2}=1+\frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}.$$

6. Imaginary Numbers

The **imaginary unit**, upon which the *complex number field* is built, is defined by

$$i=\sqrt{-1}.$$

$$i^1=i$$
$$i^2=?$$
$$i^3=?$$
$$i^4=?$$
$$i^5=?$$
$$\vdots=\vdots$$

8. So what simplified value represents i^{137}?

9. What about $i^{3,987,243}$?

For much more see "Existence of $\sqrt{-1}$" in Discovering the Art of Mathematics - Reasoning, Proof, Certainty and Truth.

7. Galileo Sums

$$\begin{aligned} 1 &= 1 \\ 1+3 &=? \\ 1+3+5 &=? \\ 1+3+5+7 &=? \\ \vdots &= \vdots \\ 1+3+5+7+\ldots+\boxed{\text{algebraic expression}} &= \boxed{\text{algebraic expression}} \\ \vdots &= \vdots \end{aligned}$$

10. Using square tiles or blocks, or graph paper to mimic them, can you arrange one tile of one color and three tiles of another color into an appropriate shape to illustrate the validity of the first equation?

11. Using one square tile of one color, three of another color and five of yet another color, can you arrange these tiles into an appropriate shape to illustrate the validity of the second equation?

12. Show how this process can be continued to provide a *Proof Without Words* for the general result.

This example has critical historical significance. In his inclined plane experiments, **Galileo** (; -) showed successfully that the distance a free falling body falls in successive time intervals is in the ratio of $1:3:5:7:9\cdots$. Over equal time intervals, summing these distances shows that the total distance is proportional to the square of the time.[1] I.e. this pattern is encoded in basic uniform motion.

8. Galileo Fractions

$$\begin{aligned} \frac{1}{3} &= \frac{1}{3} \\ \frac{1+3}{5+7} &=? \\ \frac{1+3+5}{7+9+11} &=? \\ \vdots &= \vdots \end{aligned}$$

A beautiful Proof Without Words for Galileo fractions can be developed from the following figure[2]:

[1]See Chapter 1 - "Galileo and Why Things Move" from The Ten Most Beautiful Experiments by George Johnson for a secondary description, "Galileo's Experimental Confirmation of Horizontal Inertia" by Stillwell Drake, *Isis*, vol. 64, no. 3, pp. 290-305 for a primary source description, and Section 4.2 - "The Studies of Galileo" in Basic Calculus: From Archimedes to Newton to Its Role in Science by Alexander J. Hahn for an introductory physics/calclus level description.

[2]From "Proof Without Words: Galileo's Ratios Revisited" by Alfino Flores and Hugh A. Sanders, *College Mathematics Journal*, vol. 36, no. 3, May 2005, p. 198.

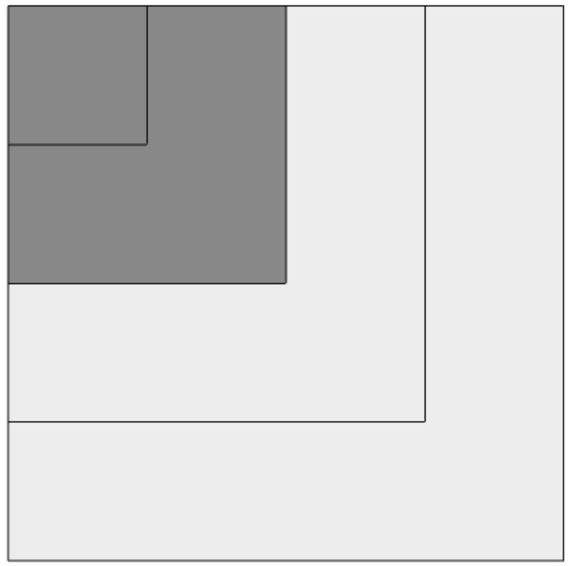


Figure 1. Proof Without Words: Galileo fractions.

9. Collatz Conjecture

There is a single pattern is consistent across and along all of these lines; the *seed value* which begins the pattern is random.

$$48 \to 24 \to 12 \to 6 \to 3 \to 16 \to 8 \to 4 \to \dots$$

$$20 \to 10 \to 5 \to 16 \to 8 \to 4 \to \dots$$

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to \dots$$

[Choose your own seed value] $\to \dots$

[Choose another seed value] $\to \dots$

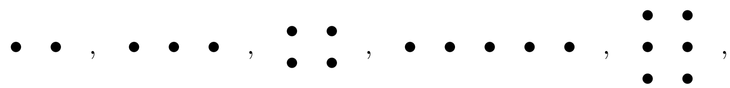
13. What conjecture might you make; i.e. do you see some meta-pattern to these patterns?
14. Try 39. What do you think now?

The meta-pattern here, and the fact that it holds for *every* seed value, is called the ***Collatz conjecture*** after **Collatz** (; -). As of this writing this is an open question, nobody knows whether this pattern continues indefinitely.

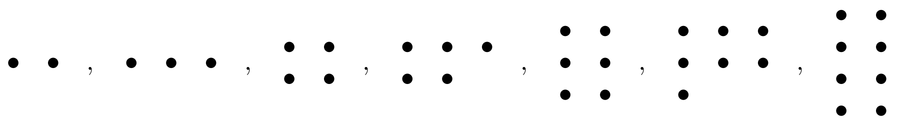
This might seem like an innocuous difficulty. However, this system is an example of the simplest possible linear dynamical system controlled by two rules. As most of the universe is essentially a dynamical system, from cellular growth which responds to its environment through to the gravitational forces controlling the dances of galaxies through the universe, it is fairly sobering that we cannot even understand how one of the simplest possible examples of such a system behaves.

10. Teacher Manual

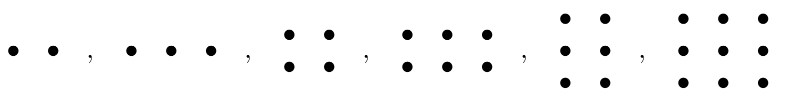
Examples of alternative patterns for 1 are:
Pattern Number 2



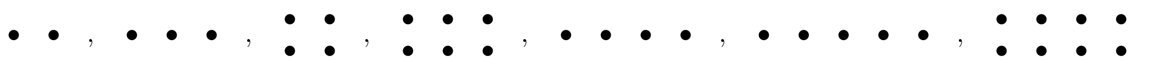
Pattern Number 3



Pattern Number 4



Pattern Number 5:



Note that there is no such thing as a "correct answer" to the pattern.

The pattern that gives rise to the title of this subsection is the representation of the consecutive integers represented in a rectangular array in which the dimensions of the array are as close to one another as possible. For example, 9 is a 3 by 3 array. All primes will have a single row. All composites more than one row. It is interesting to note that when searching for prime factors of a number n it is sufficient to search up to $\sqrt{n}$ as any factorization of n will have to include one factor $\geq \sqrt{n}$ and the other $\leq \sqrt{n}$. This pattern was chosen to show this visually.

The pattern in 2 is a way to motivate the fact that the product of a positive number and a negative number is negative. This is something almost all students have simply accepted on faith. They are quite interested to see it arise out of a pattern, where is seems natural.

The continuation, which builds on the original pattern, allows them to see why the product of two negatives naturally is a positive.

In 3 none of the patterns continue indefinitely. Calculators are of limited use as they quickly run out of digits *and* they obscure the real reason for the pattern. Computing 1111×1111 by hand shows clearly why the pattern occurs and indicates where it will fail.

For 4 the pattern is the day of the week of the 7th of February falls on for the next many years. Of course, any fixed date will give rise to a similar pattern.

The pattern in 6 offers a wonderful opportunity to consider rules for exponents. Students generally break things up into simpler pieces. So it is not unusual for them to write:

$$i^{137} = \underbrace{i \times i \times i \times \ldots \times i}_{137} = \underbrace{(i \times i \times i \times i) \ldots (i \times i \times i \times i)}_{39} \times i$$

This can be much more nicely written as

$$(i)^{136} \times i = \left(i^4\right)^{39} \times i = (1)^{39} \times i = i,$$

and we see a nice rationale for utility of the rules for exponents.

CHAPTER 3

Pick's Theorem

Not a great deal is known about **Georg Alexander Pick** (Austrian Mathematician; 1859 - 1942), but it seems he did play some nontrivial role in the early career of Albert Einstein:

> Pick was the driving force behind the appointment and Einstein was appointed to a chair of mathematical physics at the German University of Prague in 1911. He held this post until 1913 and during these years the two were close friends. Not only did they share scientific interests, but they also shared a passionate interest in music. Pick, who played in a quartet, introduced Einstein into the scientific and musical societies of Prague. In fact Pick's quartet consisted of four professors from the university including Camillo Krner, the professor of mechanical engineering.[1]

Suggestions have been made that he played a direct role in the development of Einstein's *general relativity theory* as Pick introduced him to some of the essential work in differential geometry of the time.[2] While Einstein was able to emigrate to the United States through a position at Princeton University, Pick could not avoid the Nazis. He was sent to the Theresienstadt concentration camp where we perished on 26 July 1942.

Pick is largely remember for a beatiful geometric result he discovered in 1899, but was not widely known until it was publicized in the book Mathematical Snapshots by Steinhaus three-quarters of a century later.

Big Question - We would like to determine the areas and perimeters of GeoBoard polygons. Is there a way to do this simply by counting pegs?

1. Suppose you happened upon a real, two-dimensional object and you needed to determine its perimeter. What would you do?
2. Suppose now that you had to determine the area of the object. What would you do?
3. Which is easier to determine, the perimeter or the area? Does it matter what the polygon is or is there a general rule? Explain.

A **Geoboard** is a board of pegs arranged in a regular, rectangular arry around which rubber-bands can be placed to create geometric polygons. Several images of Geoboards in action are shown in Figure 2. If you have access to one, it may be helpful below. Alternatively, you can use an online version (e.g. `http://nlvm.usu.edu/en/nav/frames_asid_277_g_1_t_3.html?open=activities\&from=topic_t_3.html` or `http://www.mathplayground.com/geoboard.html`) or simply use dot-paper or graph-paper.

On a Geoboard the unit length is the distance between adjacent vertical (or horizontal pegs). The unit area is the area of a square all of whose boundary pegs are pairwise adjacent.

4. Make a dozen or so rectangles on a Geoboard. For each determine the area and the perimeter.
5. Is there anything common about the area measurements or the perimeter measurementss? Is this surprising?

[1] `http://www-history.mcs.st-and.ac.uk/Biographies/Pick.html`
[2] See e.g. `http://en.wikipedia.org/wiki/Georg_Alexander_Pick`.

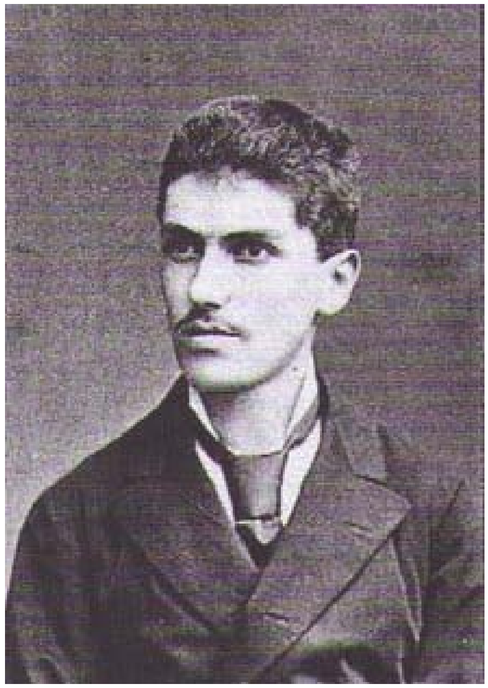


FIGURE 1. Georg Alexander Pick.

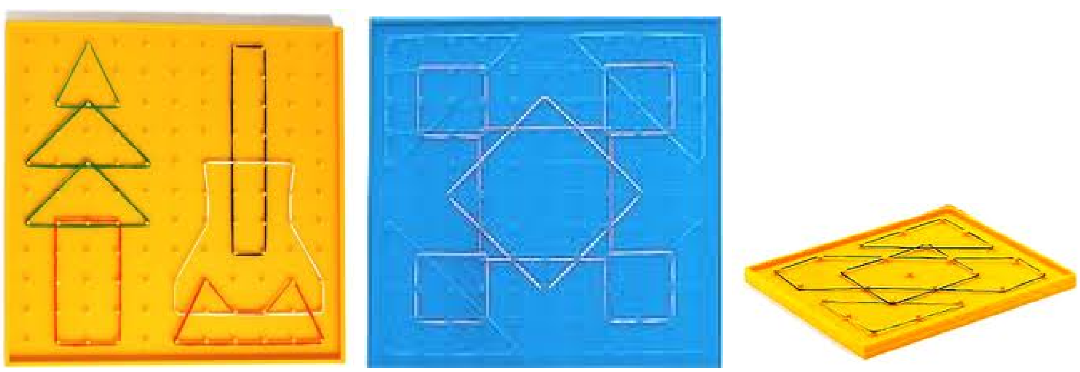


FIGURE 2. Geoboards.

6. For each of these same rectangles, record the number of ***boundary pegs***, i.e. those that are touched by the rubber-band, and the number of ***interior pegs***, i.e. those that are within the confines of the polygon created by the rubber-band.
7. Do you see a relationship which allows you to predict the area of a rectangle by knowing the number of boundary and interior points? Explain.
8. Do you see a relationship which allows you to predict the perimeter of a rectangle by knowing the number of boundary and interior points? Explain.
9. Do you think that any pattern you found above will hold for all rectangles? Either prove your result or find a counter-example to your conjecture.

10. If you have found a positive result, can you extend this result to other non-rectangular GeoBoard polygons? If not, explain. If so, describe carefully the class of objects for which the result holds and any limitations.

11. Now make a dozen or so triangles. For each, determine the area. Carefully describe your method and how you know that it is correct. (If you use a formula, you must justify the formula *and* each application of it.)

12. For each of your triangles, determine the perimeter. (Hint: You may need to remind yourself of the *Pythagorean theorem*.)

13. Is there anything common about the area measurements measurements of the triangles? Is this surprising?

14. Is there anything common about the perimeter measurements? Is this surprising?

15. How do these answers compare to what you reported in Investigation **5**?

16. Is there a way to predict the perimeter of the triangles by knowing the number of boundary and interior points? Explain.

17. Is there a way to predict the area of the triangles by knowing the number of boundary and interior points? Explain.

18. For at least one of perimeter and area you should have found a positve result. Describe this result precisely with an algebraic formula.

19. Do you think that this result will hold for other GeoBoard polygons than rectangles and triangles? Explain.

A ***heuristic argument*** is one that uses analogy, insight or organized intuition to explain a result.

20. For a heuristic argument why the positive result may hold, which is more important to the measure in question (area or perimeter), boundary or interior points? I.e. if you wanted to increase the measure in question (area or perimeter), do boundary or interior points make a greater contribution? How much greater?

If you relied on the area formula for triangles that you learned in high school it may have limited your choice of triangles above. In fact, to find the area of a random triangle theoretically is fairly complicated unless the triangle is set up so the base and height are nearly obvious. Given the Cartesian (i.e. (x, y)) coordinates of each of the vertices A, B, C of the triangle the area can be computed using the ***surveyor's formula***

$$A = \frac{1}{2}\left[x_A y_B + x_B y_C + x_C y_A - x_B y_A - x_C y_B - x_A y_C\right],$$

which is also known as the *shoelace formula* or *Gauss' area formula* after **Carl Freidrich Gauss** (German mathematician; -). If the lengths of each side of the triangle are known, the area can be computed using ***Heron's formula***

$$A = \sqrt{\frac{l_A + l_B + l_C}{2}\left(\frac{l_A + l_B + l_C}{2} - l_A\right)\left(\frac{l_A + l_B + l_C}{2} - l_B\right)\left(\frac{l_A + l_B + l_C}{2} - l_C\right)},$$

where l_A, l_B, l_C are the lengths of the triangle's sides, which was discovered by **of Alexandria Heron** (Greek mathematician and engineer; 10 AD - 70AD).

Triangular area doesn't seem quite as easy as the formulas makes it out to seem, does it?

21. Determine the area of the area of the Geoboard polygon on the left of Figure 3. Explain your reasoning in detail.

22. Determine the area of the area of the Geoboard polygon in the center of Figure 3. Explain your reasoning in detail.

23. Determine the area of the area of the Geoboard polygon on the right Figure 3. Explain your reasoning in detail.

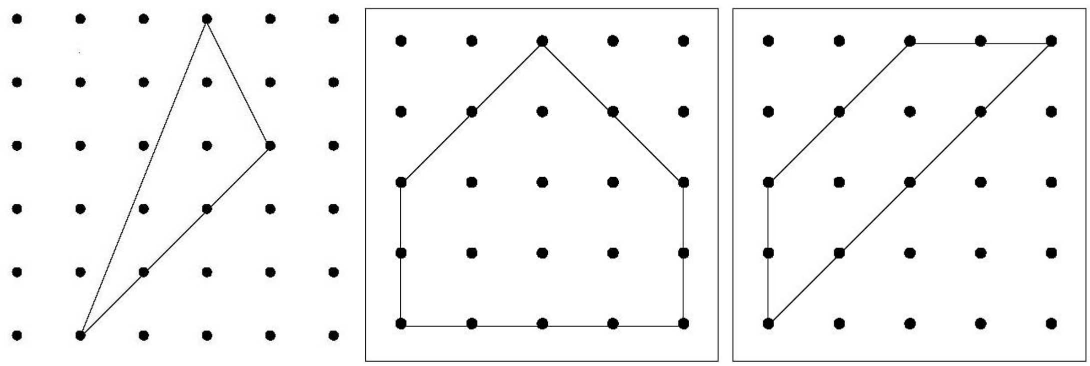


FIGURE 3. Geoboard polygons.

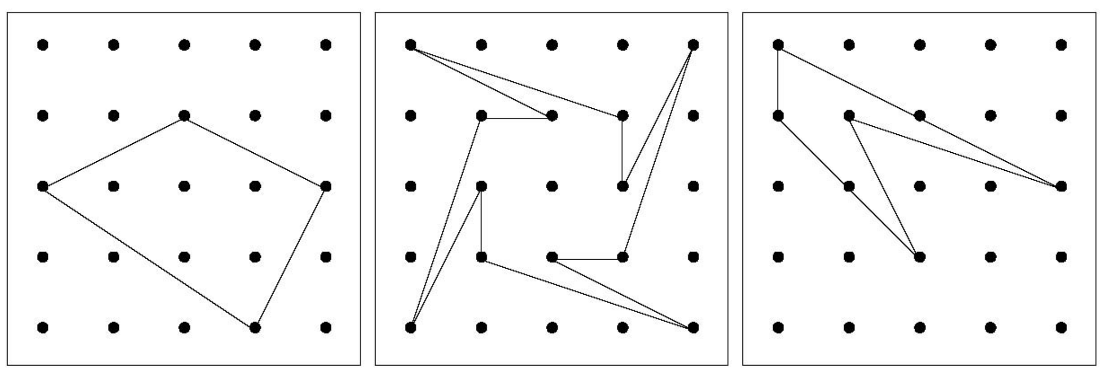


FIGURE 4. Geoboard polygons.

24. Determine the area of the area of the Geoboard polygon on the left of Figure 4. Explain your reasoning in detail.
25. Determine the area of the area of the Geoboard polygon in the center of Figure 4. Explain your reasoning in detail.
26. Determine the area of the area of the Geoboard polygon on the right Figure 4. Explain your reasoning in detail.

The positive result that you found inInvestigation **18** is called ***Pick's theorem***.

27. Does Pick's theorem hold for the polygons in Figure 3? Explain.
28. Does Pick's theorem hold for the polygons in Figure 4? Explain.
29. Use Pick's theorem to explain why the observation in Investigation **13** is valid.
30. Do you think that Pick's theorem will hold for all Geoboard polygons? Explain.

1. Applications of Pick's Theorem

Whether we and our politicians know it or not, Nature is party to all our deals and decisions, and she has more votes, a longer memory, and a sterner sense of justice than we do.
Wendell Berry (American Poet; -)

Documenting the amount of ice at the Earth's poles is of fundamental importance in our understanding of the causes and impact of global warming. Figure **??** shows the extent of the polar sea ice in 1979 and 2012.

2. Generalizing Pick's Theorem

In May, 1991 the National Council of Teachers of Mathematics published an article in its journal *Mathematics Teacher*, a journal for grade 6 - 14 mathematics teachers. The title of the article is "Pick's Theorem Extended and Generalized."

Many Geoboards have a square grid on the front and a different grid on the back. Typical alternatives are equilateral triangle lattices and hexagonal (like honeycomb) lattices. The article shows how Pick's formula can be adapted to hold for these other shapes as well.

The abstract of the article reads:

> Our mathematics teacher set us an assignment to investigate Pick's theorem for square lattices. When I finished early, he suggested that I extend the theorem to cover triangular lattics, which I was able to do without too much trouble. The next step was obvious - could I extend the theorem to all lattices? The hexagonal lattice proved difficult, but I worked on it that night and the next day. While I was walking home from school, I realized that if I introduced a third variable for the type of lattice, I had a formula to cover all lattices. I arrived at the formula through a process of seeing patterns and guessing and checking, and I am not, at this stage, able to give a more mathematical proof.

Typically mathematical articles are not published without containing "a more mathematical proof." But this was not a typical article. The author here, Christopher Polis, was *an eighth grade student* when he discovered how to extend Pick's theorem to general Geoboard lattices of different shapes!

3. Proving Pick's Theorem

> In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels [s]he ought to know why it appears.
>
> **W. W. Sawyer** (; -)

Pick's theorem is surprising in its simplicity. But why does it work? Mathematicians are not satisfied that a result is true until there is a proof. Not compelled that a proof is really necessary after all of the evidence supporting Pick's theorem above?

31. For the Geoboard polygon in Figure 6, determine the area, number of interior points and number of boundary points.

32. Does Pick's theorem hold for this figure?

33. For the Geoboard polygon in Figure **??**, determine the area, number of interior points and number of bounday points.

34. Does Pick's theorem hold for this figure?

35. Can you explain why these *counterexamples* arise, at least intuitively?

36. Do these examples bother you? Explain.

In Investigation **20** you were asked to think about how adding boundary or interior points contributed to an increase in area. Let's try to apply this heuristic to Pick's formula more formally.

37. On graph paper draw a Geoboard rectangle. Find the area, number of boundary points and number of interior points.

38. Draw vertical and horizontal dotted/colored lines exactly between each of the lines on your graph paper.

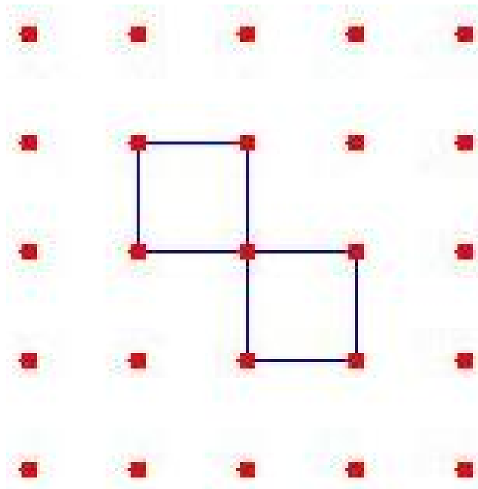


FIGURE 5. Does Pick's theorem hold?

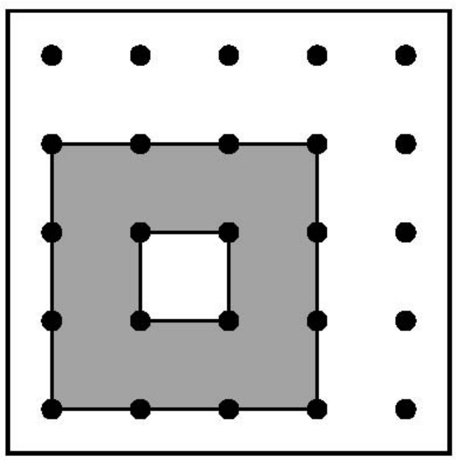


FIGURE 6. Does Pick's theorem hold?

39. Explain why your new figure shows that each interior point contributes exactly one unit of area to the total.

40. Does your figure show that each boundary point contributes exactly one-half unit of area? Explain.

41. Use your figure to prove Pick's theorem for rectangles.

So here's the question - Can we extend this to general Geoboard polygons? It turns out we can.[3]

For each point, boundary and interior, of a Geoboard figure, define the **visibility measure** at that point to be the relative percentage you can see into the interior of the Geoboard figure from that point.

[3]This approach is from "Pick's Theorem Revisited" by Dale E. Varberg, *American Mathematical Monthly*, vol. 92, no. 8, October 1985, pp. 584-7.

For example, for the corner of a rectangle the visibility measure is $\frac{1}{4}$. For any non-corner peg on the boundary of a rectangle the visibility measure is $\frac{1}{2}$.

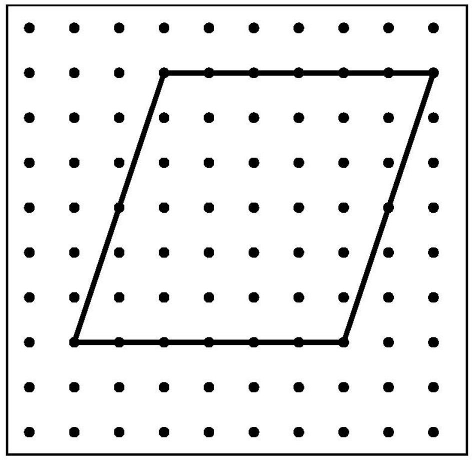


FIGURE 7. Geoboard quadrilateral.

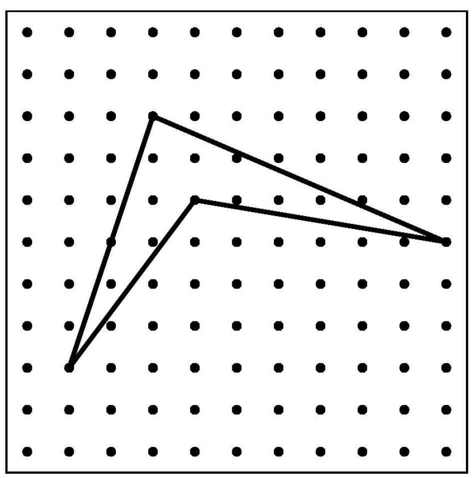


FIGURE 8. Geoboard quadrilateral.

42. Find the visibility measure of each interior and boundary point in Figure 3.
43. For each Geoboard figure, add up all of the visibility measures. What do you notice?
44. Find the visibility measure of each interior and boundary point in Figure 4.
45. For each Geoboard figure, add up all of the visibility measures. What do you notice?
46. Find the visibility measure of each interior and boundary point in Figure 7.
47. Find the area of the Geoboard polygon in Figure 7. How does the area compare to the sum of the visibility measures?
48. Find the visibility measure of each interior and boundary point in Figure 8.

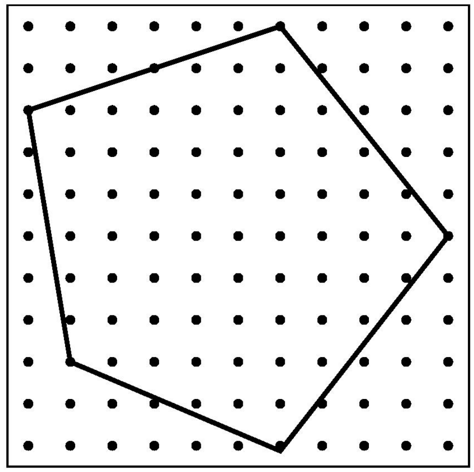


Figure 9. Geoboard pentagon.

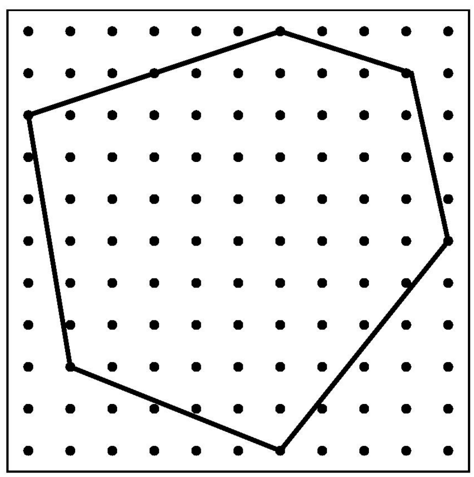


Figure 10. Geoboard hexagon.

49. Find the area of the Geoboard polygon in Figure 8. How does the area compare to the sum of the visibility measures?
50. Find the visibility measure of each interior and boundary point in Figure 9.
51. Find the area of the Geoboard polygon in Figure 9. How does the area compare to the sum of the visibility measures?
52. Find the visibility measure of each interior and boundary point in Figure 10.
53. Find the area of the Geoboard polygon in Figure 10. How does the area compare to the sum of the visibility measures?
54. Find the visibility measure of each interior and boundary point in Figure 11.
55. Find the area of the Geoboard polygon in Figure 11. How does the area compare to the sum of the visibility measures?

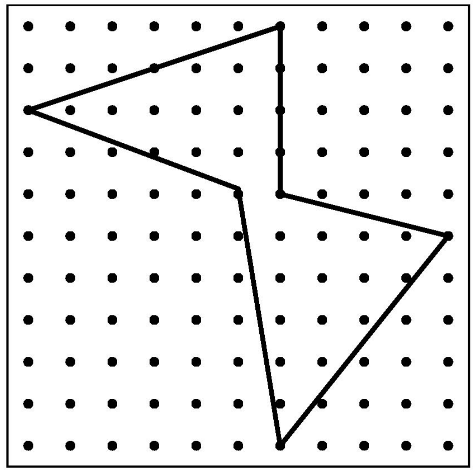


FIGURE 11. Geoboard hexagon.

56. You should notice that certain types of interior and boundary points have predictable visibility measures. What are they?
57. Do you think that the relationship between the area and the sum of the visilibity measures will hold for all Geoboard figures? Why?

For any Geoboard polygon, S, define the **measure** $M(S)$ to be the sum of the visibility measures of the interior and boundary visibility measures.

58. You've already proven that $M(S) = A(S)$ whenever S is a rectangle. Explain.
59. Suppose the Geoboard polygon S is composed of two other Geoboard polygons T and R who may share boundaries but are otherwise non-overlapping. Prove that $M(S) = M(T) + M(R)$.[4]
60. Explain why, in the context of the previous problem, $A(S) = A(T) + A(R)$.
61. Explain why $M(S) = A(S)$ must be true for right, Geoboard triangles.
62. Explain why it now follows that $M(S) = A(S)$ for all Geoboard triangles.
63. Can all Geoboard polygons be decomposed into Geoboard triangles? Explain.
64. Combine these results to explain why it follows that $M(S) = A(S)$ for all Geoboard polygons S.

We've now found an alternative way to measure areas of Geoboard polygons. How's this help us prove Pick's theorem? With rectangles and in Investigation **??** the visibility measures of points were very different depending on the type of point that we had.

65. For a given Geoboard figure, if you add together all of the visibility measures that equal 1, what is the sum equal to? (Equal to in relation to terms that are essential to Pick's formula.)
66. For a given Geoboard figure, if you add together all of the visibility measures that equal $\frac{1}{2}$, what is the sum equal to? (Equal to in relation to terms that are essential to Pick's formula.)

The angle formed on the inside of a Geoboard polygon where two straight line segments come together is called the ***interior angle*** of the polygon.

67. For several triangles you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?
68. For several quadrilaterals you have already considred, determine each of their interior angles and the sum of their interior angles. What do you notice?

[4]This property is called ***subadditivity***.

69. For several pentagons you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?

70. For several hexagons you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?

71. Make a conjecture about the sum of the interior angles of a polygon with n sides.

A ***simple polygon*** is a polygon whose sides are connected to each other one after another until *closed* without *intersecting*.

72. Figure **??** provides a "Proof Without Words" that the sum of the exterior angles in a simple polygon with n sides agrees with your answer above. Explain this proof. Illustrate how it works with several other Geoboard polygons you have considered above.

73. Do you find this proof to be compelling? Explain.

74. Does your formula for the sum of the interior angles apply to the Geoboard polygon in Figure 6? So is this a counterexample to our angle sum conjecture?

75. Does your formula for the sum of the interior angles apply to the Geoboard polygon in Figure **??**? So is this a counterexample to our angle sum conjecture?

76. Complete the proof of Pick's theorem by determining the sum of the visibility measures of the points whose visibility measures differ from 1 and $\frac{1}{2}$.

Story about the kid. Amateur mathematicians.

Lorenze and weather!!! Is all of this in the Reasoning book? It should be. Along with the Want of a Nail stuff.

4. Teacher's Guide

The result that Boundary = Perimeter in 9 does indeed hold for all rectangles. There are several proofs, all of which are of interest.

One approach is simply to disconnect the boundary at a single peg. The boundary can be straightened out and it is clear that there is a one-to-one correspondence between each "peg" and unit of perimeter.

An algebraic approach is to notice that along each edge the number of pegs is one more than the length of that edge. So if p_{base} and p_{height} are the number of pegs along the base and the height of the rectangle, then $p_{base} = b+1$ and $p_{height} = h+1$. The perimeter of the rectangle is $P = 2b+2h$. We can add the pegs along each side of the rectangle, noting that this will count each corner twice, so that the total number of pegs is $p_{boundary} = p_{base}+p_{height}+p_{base}+p_{height}-4 = (b+1)+(h+1)+(b+1)+(h+1)-4 = 2b + 2h + 4 - 4 = 2b + 2h = P$.

An inductive approach is also very interesting. The result is clearly true for a unit square. To make a 1×2 rectangle from the unit square, move the right side over one unit to the right. To complete the rectangle, two units of perimeter will need to be added and two more boundary pegs will need to be added. So we still have $P = Boundary$. In fact, if we move *any* edge of a rectangle one unit in a direction perpendicular to its length to make a larger rectangle, we will have to add exactly two new units of perimeter and two new boundary pegs. This enables us to create any rectangle from the initial square while insuring at each stage that $P = Boundary$.

Polygons like triangles certainly do not have $P = Boundary$ and students are quick to assume that the result only holds for rectangles in . However, this result can naturally be extended to figures such as the one pictured in Figure 12. Caution must be used in generalizing too far however. Pick's theorem is equivalent to Euler's theorem for polyhedra (put ref here??), so all of the counterexamples and refutations that occur there also arise here. For example, the polgon pictured in Figure 13 does not satisfy $P = Boundary$, nor does it satisfy Pick's theorem.

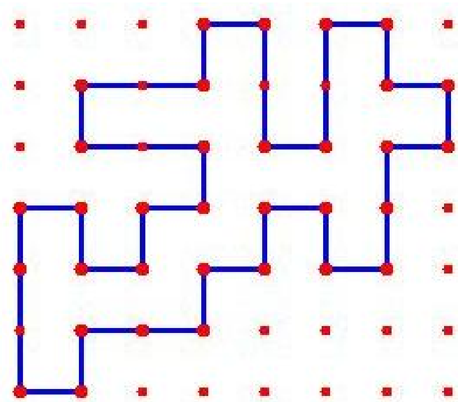


FIGURE 12. A non-rectangular Geoboard polygon for which $P = Boundary$.

As soon as triangles are introduced there will be questions of wheather certain pegs are boundary pegs or not. For example in Figure 14 it is clear that there are, respectively, none and one boundary points on the hypotenuses (not including the vertices). It is *not* so clear on an actual Geoboard. The thickness of the rubber bands and the thickness of the pegs make several more points seem candidates for boundary points. This is a wonderful opportunity for students to think back about properties of

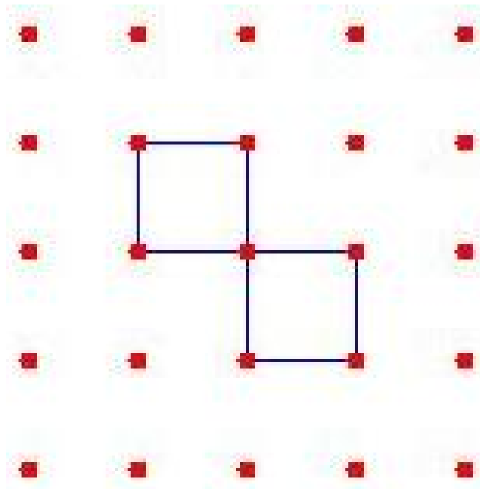


FIGURE 13. A counterexample to $P = Boundary$ for *rectilinear Geoboard polygons and* Pick's theorem.

lines, including slope. (For example, see the video `http://artofmathematics.westfield.ma.edu/maa-focus/posters/BoundaryPointDiscussion.mov`.

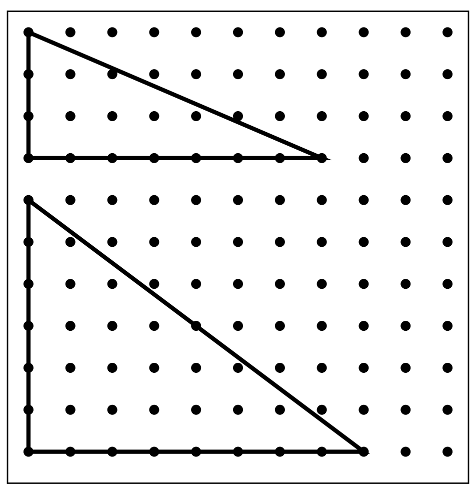


FIGURE 14. What constitutes a boundary peg on a Geoboard.

In trying to discover Pick's theorem the students often make very interesting geometric patterns (e.g. rectangles that are 3 by 4, then 3 by 5, then 3 by 6, etc., and then find interesting patterns. But not the more general relationship discovered by Pick. Often it helps to ask them how they are organizing their data. If it is just in the order they created the geometric polygons, other orders can help them. For example, consider all of the geomteric polygons that have the same number of boundary pegs. How does the area vary then? Or consider all of the geometric polygons that have the same number of interior pegs. How does the area vary then?

CHAPTER 4

The Use of Patterns and Language in the Creation of Powerful Number Systems

> Science is the attempt to make the chaotic diversity of our sense-experiments correspond to a logically uniform system of thought.
>
> **Albert Einstein** (; -)

> We encounter patterns all the time, every day: in the spoken and written word, in musical forms and video images, in ornamental design and natural geometry, in traffic patterns, and in objects we build. Our ability to recognize, interpret, and create patterns is the key to dealing with the world around us.
>
> **Margorie Senechal** (; -)

[All of these things revolve around language. This is very important and might provide a good context. The only issue is to keep it really closely related to patterns. The bridge between the language and patterns must be clear or this will be out of place here. One of the key things is that the patterns are what are used to extend the language. It is the power of the language that these extensions continue to have such a profound power. Roman numerals did not have that, for example. Nor did simple fractions as incommensurability shows.]

Brief discussion about quantity at the outset? Studies about babies and animals being able to recognize differences in small quantities.

Counting is a much bigger leap. Thing about dolphins being able to do simple arithmetic. But it is clear that humans have gone tremendously far beyond that. Look at the language we have developed to count, measure, and communicate about quantity. [Give GDP numbers, angular measurements, building measurements, ISO measurements, etc.]

This language seems natural to us in many ways as we have grown up with it. But it is not natural at all. It is a phenomenal intellectual achievement which is largely quite modern.

Humans have been on the planet for ???? years. [Provide a brief discussion here about the relative length of time that humans have used language to communicate. Then segue into the profoundly shorter length of time there has been an appropriate language for mathematics - even just in terms of numeration. Then Sand Reckoner. Then no zero until sixteenth century.]

Seen in this light, our base-ten system of numeration and our use of exponents and scientific notation are profound achievements.

Our need for using these tools in sophisticated ways varies wildly depending on our occupations and other vocations. But it seems unfortunate that after so much exposure to this language in school the profundity of this language are totally lost on so many. Here we will look at a few of these high points.

Base-Ten System and Huge Number Names

[Links to scientific notation - names versus symbols.]

Need to say what “base-ten” numbers are so they can start the problems below.

Base-Two Number Trick

1. Investigations

1.1. Names of Numbers. For each of the numbers whose name in English is given below, write the number named as a base-ten number:

77. Nineteen thousand, four hundred sixty-five.
78. Three hundred fifty two thousand, eight hundred nineteen.
79. Seventeen million, forty three thousand, five hundred eighty two.

When writing numbers in English one uses commas only to separate words as you would when writing the digits - only in groups of three. The word "and" is used to indicate where a decimal point goes.

For each of the numbers written below in base-ten, give their English name:

80. 784.
81. 562,978.
82. 6,587,581.
83. 5,914,490,937.

We can write large numbers, in fact as large as we desire, because the base-ten Hindu-Arabic number system that we use is positional. We do not need to adapt this system in any way, we just fill in appropriate digits to build larger and larger numbers. Such a number system is dramatically more powerful and flexible than the Greek, Roman, and Egyptian systems which were the dominant systems through much of recorded Western history.

This system that we generally take for granted is a profoundly powerful human language.

You should remember how this positional number system works. 5,327 is five thousands, three hundreds, two tens, and 7 ones, written in ***expanded notation*** as:

$$5 \times 1000 + 3 \times 100 + 2 \times 10 + 7 \times 1.$$

The *digits* are always 0 - 9 and the critical bases are ..., 1000, 100, 10, and 1.

For positive integers a and n we define a to the **power** n by $a \times a \times \cdots \times a$ where there exactly n factors of a in the product. The number a is called the **base** and the number n is called the **exponent**. It is then natural to call the numbers $a, a^2, a^3, \ldots$ the **powers of** a.

84. Is 100 a power of 10? Explain.
85. Is 1000 a power of 10? Explain.
86. Is 10000 a power of 10. Explain.
87. You should see a pattern forming in Investigations **84-86**. Use this pattern to write the number $1\underbrace{000\ldots000}_{n}$ as a power of 10 where the number has n zeroes following the lone digit 1.
88. Use Investigation **87** to explain why it is appropriate to call our number system the "base-ten" number system.

We can now simplify the expanded notation using what we have learned about powers of ten. So, for example, we can write $5,327$ as $5 \times 10^3 + 3 \times 10^2 + 2 \times 10^1 + 7 \times 10^0$ where, for the moment, we have defined $10^0 = 1$. (Shortly we will see that this really is a natural consequence of a pattern.)

89. Write the number in Investigation **80** in expanded notation using powers of ten.
90. Write the number in Investigation **81** in expanded notation using powers of ten.
91. Write the number in Investigation **82** in expanded notation using powers of ten.
92. Write the number in Investigation **83** in expanded notation using powers of ten.

Because successive powers of ten are a factor of ten larger than the power which precedes it we have a powerful tool to study things that exist on massive scales - like our universe. In their wonderful book

Powers of Ten authors **Philip Morrison** (American Physicist and Author; 1915 - 2005)[1] and **Phylis Morrison** (American Teacher, Educator, and Author; - 2002) provide a tour of the universe by starting at the edge of our local cluster of galaxies and with each successive page moving our viewpoint a factor of ten closer. After some 25 pages we see we have been focusing on a couple lying on a blanket at a city park in Chicago, Illinois. Not stopping there, the photos continue to move ten times closer, eventually reaching the sub-atomic particles that make up the DNA of one of these people. Subsequent movies, flip-books, screen savers, and interactive Internet sites immortalize this powerful idea.[2]

The American Museum of Natural History in New York, New York integrates these ideas into a spectacular installation called "Scales of the Universe." At the center of this installation housed in the Rose Center for Earth and Space is the Hayden Planetarium - a 150 foot tall sphere which houses a full IMax theatre in the top half and an interactive tour on the bottom half. Spiraling around the Hayden Sphere is "Scales of the Universe" - a walkway through the sizes and scales of the universe. Instead of using visual images like Powers of Ten, it uses physical models which are successively compared to the massive Hayden Sphere which hangs right in front of your view to help you understand the awesome scale of the universe through the subatomic workings of each little piece of the universe.

What about naming really large numbers using English words?

93. Choose a number which is a four digit number when written in base-ten. Write this number in base-ten and then name this number using English words.
94. Now add a digit to the front and name this number using English words.
95. Continue adding digits one at a time to the front of your number and naming your new number using English words. Whenever possible, you need not name the number completely if a significant part of the name stays the same. Just note what part stays the same. You should continue doing this until you have a 14-digit number.
96. Suppose you were asked to add one more digit. Would you need any additional information to know how to name your number using English words? Explain.
97. Suppose you were asked to add four more digits. Would you need any additionally information to know now to name your number using English words? Explain.

The terms million, billion, trillion, quadrillion, quintillion, sextillion, septillion, octillion, and nonillion were introduced by **Nicholas Choquet** (; -) in 1484 and appeared in print in a 1520 book by **Emile de la Roche** (; -). The meanings of these words were subsequently changed and there continue to be linguistic debates and discrepancies about the names of large numbers.

Nonetheless, there is a fascination with naming large numbers. Mathematicians have continued to develop different schemes. In 1996 **Allan Wechsler** (; -), **John Horton Conway** (; -), and **Richard Guy** (; -) proposed a system that can be extended indefinitely to provide an English word name for *any* number! We will consider this naming scheme now.

98. Complete the chart below, using your knowledge of prefixes and patterns to help you establish a pattern that gives meaning to the terms you are not certain of:

[1] More info here about him? E.g. SETI patriarch...

[2] This is also in The Very Large chapter. How do we reference it here again?

Name of Number	Number in Base-Ten	Number as Base-Ten Exponent
Thousand	1,000	
Million	1,000,000	
Billion		
Trillion		
Quadrillion		
Quintillion		
Sextillion		
Septillion		
Octillion		
Nonillion		

99. Do these names give you what you needed to positively answer Investigation **96** and Investigation **97**? Explain.

100. Using these names, how high can you count before you will not be able to name a specific number? Describe this number.

Our goal is to use patterns in this chart to expand our linguistic ability to name increasingly larger numbers.

101. Add two more columns to your table from Investigation **98** - the columns which have been started below. Complete the columns in the natural way.

Prefix of Number	Name Ordinal illion
None	zeroeth
mi	first
bi	second

102. Find patterns in the table that allow you to determine the 10^{th}, 13^{th}, and 21^{st} illion as base-ten exponents.

103. Extending Investigation **102**, the n^{th} illion is equal to $10^{?}$.

104. Conversely, what illion is 10^{57}? 10^{219}? 10^{399}?

It is clear how our base-ten exponents and ordinal illions can continue indefinitely. What we need to extend the English names of the numbers is more prefixes. Wechsler, Conway and Guy provided a way to extend these prefixes indefinitely. Their scheme relies on the following prefixes:

	Units	Tens	Hundreds
1	mi	deci^{n}	centi^{nx}
2	bi	viginti^{ms}	ducenti^{n}
3	tre*	triginta^{ns}	trecenti^{ns}
4	quad	quadraginta^{ns}	quadringenti^{ns}
5	quint	quinquaginta^{ns}	quingenti^{ns}
6	se*	sexaginta^{n}	sescenti^{n}
7	sept*	septuaginta^{n}	septingenti^{n}
8	oct	octoginta^{mx}	octingenti^{mx}
9	non*	nonaginta	nonagenti

The prefixes are attached in the order of units, tens, and then hundreds for historical reasons.

Like many prefixes, slight modifications are necessary depending on the context in which they are used. Modifications are needed for the four units marked with $\star$ above. They are as follows:

- "tre" becomes "tres" when used directly before a component marked with an s and when used directly before illion it becomes "tr" as in "trillion".
- "se" becomes "sex" when used directly before illion or a component marked with an x and becomes "ses" when used directly before a component marked with an s.

- "sept" and "non" become "septem" and "novem" when used directly before a component marked with an m and become "septen" and "noven" when used directly before a component marked with an n.

Examples:

- quintdecisescenti is 615 so quintdecisescentillion is the 615^{th} illion, 10^{1848} as a base-ten exponent.
- septemoctigenti is 807 so septemoctigentillion is the 807^{th} illion, 10^{2424} as a base-ten exponent.

105. Write the name and the base-ten exponent which represent the 237^{th} illion.
106. Write the name and the base-ten exponent which represents that 649^{th} illion.
107. Name the numbers 10^{57}, 10^{219}, and 10^{399} which you considered in Investigation **104**.
108. Name and write as a base-ten exponent the number that is the largest number that can be written in this naming scheme.

We wanted to be able to name arbitrarily large numbers. What Wachsler, Conway, and Guy did to surpass the limit in Investigation **108** was to extend their numbering scheme in blocks of 1,000 with "illi" as a separator and "nil" representing 0.

Examples:

- millinillitrillion is the $1,000,003^{\text{rd}}$ illion.
- trecentillinillioctillinonagintillion is the $300,000,008,090^{\text{th}}$ illion.

109. What illion and what base-ten exponent has the name novemsexagintatrecentillitresquadragintasescenti?
110. What illion and what base-ten exponent has the name trestrigintillinillimicentillion?
111. Write the name and the base-ten exponent which represents the $42,903,271^{\text{st}}$ illion.
112. Name the number $10^{24,921,846}$.

So far the numbers we have named have only had 1 as a leading digit and we have been jumping from one illion to the next skipping over all the intermediary numbers. But it is easy to fill in this gap if we simply use our everyday knowledge of naming numbers as illustrated in Investigations **80-83**.

113. Name the number $27,\underbrace{000,\ldots,000}_{36}$.
114. Name the number $400,\underbrace{000,\ldots,000}_{150}$.
115. Name the number $56,320,\underbrace{000,\ldots,000}_{318,174,639}$.

Essay 1 I'm thinking that it would be nice to have a brief essay question here that encouraged students to reflect on what was just learned. But I am not thinking of a good one. So, for this question: a) Think up a brief essay prompt related to the work above, and, b) Answer your question.

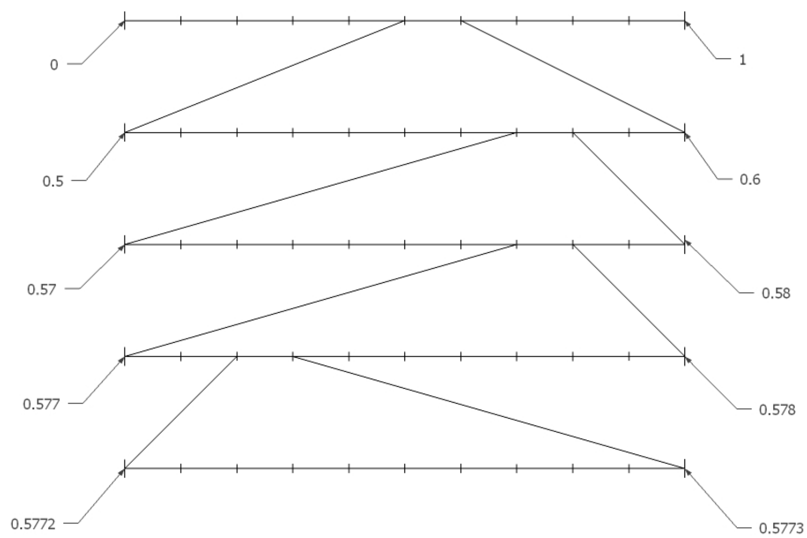


FIGURE 1. Magnifying part of the real number line.

1.2. 0.999999 . . . **and** 1**.** The set of *real numbers* contains all of the numbers that we work with in ordinary life: $3; 271; 1.5;\ 199.99; 1,000,000, \pi, e$, etc. One way to think of the real numbers is as what are necessary to measure lengths. For example, π is the length of the perimeter (aka the *circumference*) of a circle of radius $r = \frac{1}{2}$.

In everyday usage we generally represent real numbers using the base-ten system considered above. Above we only utilized whole numbers, here we will use decimal digits as well.

So what do decimal digits tell us? One way to think of them is as an ***address*** of where a given number lies on a number line. Illustrated in Figure 1 is what one would see if one repeatedly magnified a portion of the number line, with the location of several real numbers labelled.

116. Label each of the division marks in the original interval $[0, 1]$ in Figure 1.
117. Label each of the division marks in the first magnified interval $[0.5, 0.6]$ in Figure 1.
118. Why are each of the intervals divided into ten equal subintervals?
119. If you are given the decimal representation of a real number, what does each individual digit tell you about its location in the appropriately subdivided interval? Explain.
120. The magnifications in Figure 1 help us begin to locate the important ***Euler-Mascheroni constant***,[3] whose decimal expansion begins 0.577215664901532, on the number line. Draw a figure which continues the illustration in Figure 1 through six more magnifications.

Here and below when we write 0.999999 . . . we mean the infinitely repeating decimal all of whose digits are 9. Sometimes this number is written compactly as $0.\overline{9}$. Because we will be doing arithmetic and algebra with this number we find it more useful to use the notation with the **ellipsis** . . .

121. Illustrate the location of 0.999999 . . . as you did above for the Euler-Mascheroni constant. Use four or five magnifications. How hard would it be to continue magnifying?

[3]It is interesting to note that this important constant has been approximated to billions of decimal digits but we have no idea whether this number can be represents a fractional, *irrational*, *algebraic*, or *transcendental number*.

122. Do you believe that $0.999999\ldots$ precisely represents a definitive, fixed, specific real number? Explain.

123. Classroom Discussion: How does $0.9999999\ldots$ compare with the number 1?

124. Use long division to precisely write $\frac{1}{3}$ as a (possibly infinite) decimal. Express your result as an equation: $\frac{1}{3} = - - - - - -$.

125. Multiply both sides of your equation from Investigation **124** by 3. What does this suggest about the value of $0.999999\ldots$? Surprised?

People often object to the result in Investigation **125** because $0.999999\ldots$ and 1 appear so different. But remember, the two expressions $0.999999\ldots$ and 1 are simply symbolic representations of real numbers. And there many representations of numbers that are not unique. For example, we can write the real number 3 as $\frac{6}{2}$, $\frac{21}{7}$, $\sqrt{9}$, III, $3.\overline{0}$, or even 11_2, the base two notation that all computers use to represent the number 3. (Link to trick above.)

126. Give several real-life examples of objects that we commonly represent in different ways.

127. In thinking about $0.999999\ldots$ as a representation of a number we might know more readily in a different symbolic guise, let us use algebra to help us. Since we aren't sure of the identity of $0.999999\ldots$, let's set $x = 0.999999\ldots$ Determine an equation for $10x$ as a decimal.

128. Using your equation for $10x$ in the previous investigation, complete the following subtraction:

$$\begin{array}{r} 10x = \\ -x = 0.9999999\ldots \\ \hline = \end{array}$$

129. Solve the resulting equation in Investigation **128** for x. Surprised?

Seventh Grader Makes Amazing Discovery

New discoveries and solutions to open questions in mathematics are not always made by professional mathematicians. Throughout history mathematics has also progressed in important ways by the work of "amateurs." Our discussion of $0.9999999\ldots$ provides a perfect opportunity to see one of these examples.

As a seventh grader **Anna Mills** (American Writer and English Teacher; 1975 -) was encouraged to make discoveries like you have above about the number $0.999999\ldots$ Afterwards Anna began experimenting with related numbers on her own. When she considered the (infinitely) large number $\ldots 999999.0$ she was surprised when her analysis "proved" that $\ldots 999999.0 = -1$! She even checked that this was "true" by showing that this number $\ldots 999999.0$ "solves" the algebraic equations $x + 1 = 0$ and $2x = x - 1$, just like the number -1 does.

Encouraged by her teacher and her father to pursue this matter, Anna contacted **Paul Fjelstad** (American Mathematician; 1929 -). Fjelstad was able to determine that Anna's seemingly absurd discovery that $\ldots 999999.0 = -1$ is, in fact, true as long as one thinks of these numbers in the settings of *modular arithmetic* and *p-adic numbers*.

You can see more about this discovery in Discovering the Art of Mathematics - The Infinite or in Fjelstad's paper "The repeating integer paradox" in *The College Mathematics Journal*, vol. 26, no. 1, January 1995, pp. 11-15.

Here's an alternative way to think about the relationship between $0.999999\ldots$ and 1, one based on the theory of *limits* that underlies the almost universally accepted framework for the system of real numbers that have been precisely defined by mathematicians.

130. Evaluate the truth of the following claim:

> Unless they are equal, any two real numbers have a fixed, non-zero distance that separates them.

Explain.

131. Classroom Discussion: Is there some fixed, non-zero distance between the real numbers 0.999999... and 1?

The discussion question Investigation **1.2** is a yes or no question - these are the only possible answers.[4]

Let us start our investigations by assuming that the answer to our question is "yes", there is some fixed, non-zero distance between 0.999999... and 1.

132. Write down a *really, really small* fixed, specific non-zero number to estimate the hypothetical distance between 0.999999... and 1. Denote this distance by the Greek letter epsilon[5] which is written as ϵ.

133. Which number is closer to 1, 0.9 or 0.999999...? Explain.

134. What is the distance between 1 and 0.9?

135. Which number is closer to 1, 0.99 or 0.999999...? Explain

136. What is the distance between 1 and 0.99?

137. Which number is closer to 1, 0.999 or 0.999999...? Explain

138. What is the distance between 1 and 0.999?

139. You should see a pattern forming. Describe this pattern precisely.

140. By adding enough zeroes, you can find a number of the form 0.00...01 that is smaller than the ϵ you chose in Investigation **132**. Do so explicitly.

141. Use Investigation **140** and the pattern you described in Investigation **139** to find a number of the form 0.99...9 so that the distance from 0.99...9 to 1 is less than the number you found in Investigation **140**.

142. You should now be able to conclude that the distance from 0.999999... to 1 is less than ϵ, contradicting your choice of ϵ in Investigation **132**. Explain.

It didn't matter how small the ϵ you chose in Investigation **132** was, this process can be repeated.

143. Explain why these investigations show that there cannot be any fixed, non-zero distance between 0.9999999... and 1.

144. Explain why this proves that $0.999999\ldots = 1$ as real numbers.

This type of argument is a fairly modern one, due in large part to the work of **Augustin-Louis Cauchy** (French Mathematician; 1789 - 1857). His definition of limits in this way was the culmination of a period of great crisis in mathematics during the middle of the nineteenth century. This crisis was foretold as early as the advent of calculus when **Bishop George Berkeley** (Irish Philosopher and Theologian; 1685 - 1753) wrote about the "infidel mathematicians" and their use of "infinitesimals", saying:

> And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?

Much more about these issues are included in the companion book Discovering the Art of Mathematics - Calculus in this series.

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[4] Technically this assumption is known as the *Law of the Excluded Middle*. While there are some mathematical philosophies and systems of logic that do not include the Law of the Excluded Middle as an axiom, this law is generally accepted and we will use it freely here.

[5] This is the typical notation for this type of limit argument. Because it is so used, the great 20^{th} century mathematician **Paul Erdös** (Hungarian Mathematician; 1913 - 1996), who is quoted and referenced in many books in this series, used to call children "epsilons".

We close our comments by noting that there are different systems of numbers than the real numbers. In particular, the *surreal numbers* considered in the companion book Discovering the Art of Mathematics - The Infinite are a system of numbers that include infinitely many different infinitely small non-zero numbers. And this opens Pandora's Box right back up.

145. Have these investigations changed your answer to Investigation **1.2**? Explain.

1.3. Using Patterns to Extend Language. [Need a context. Can talk about the use of zero, negative numbers, etc. This is already in the infinite thing. How to cross-reference it here? It is important because it illustrates nicely the development of thought in mathematics. What is obvious to one generation is not to another. The great quote by Schrodinger about root 2. Sand Reckoner is a good segue. It is really just shorthand. But it becomes quite powerful and is now used in remarkably sophisticated ways. (Set the stage for Euler's formula at end.) The use of exponents in algebra is often a problematic area because one must extend fairly intuitive conventions into more abstract realms. Here we illustrate how patterns naturally explain that extension.]

146. Using the definition of powers above, describe what 2^4, 5^2 and 7^3 mean. Then convert each of these into base-ten numbers with no exponents.

147. Complete the following table by filling in five more rows with different positive integer values of m and n:

m	2^m	2^m as base-ten #	n	2^n	2^n as base-ten #	$2^n \times 2^m$ as base-ten #	Is $2^n \times 2^m$ a power of 2?
3	2^3	8	5	2^5	32	256	Yes. $256 = 2^8$.

148. Based on your table, for positive integer values of m and n is $2^n \times 2^m$ always a power of 2?

149. If you answered Investigation **148** in the affirmative, find a formula which expresses $2^n \times 2^m$ as a single power of 2.

150. Return to the definition of powers and show why the result in Investigation **149** really follows from the definition of powers. (I.e. provide a proof of the result in Investigation **149**.)

151. Repeat Investigations **147-150** for a positive integer base different than $a = 2$.

152. Will the rules for exponents you found in Investigation **149** and Investigation **151** hold for any base a ¿ 0? Explain why and how you know this.

Having determined patterns in the multiplication of powers to a common base, a natural question is whether there is a corresponding rule for division.

153. Complete the following table by filling in five more rows with different positive integer values of m and n:

m	2^m	2^m as base-ten #	n	2^n	2^n as base-ten #	$2^n \div 2^m$ as base-ten #	Is $2^n \div 2^m$ a power of 2?
3	2^3	8	5	2^5	32	4	Yes. $4 = 2^2$.

(Warning: There are some issues for $m \geq n$ that will have to be considered later.)

154. Based on your table, for positive integer values of m and n is $2^n \div 2^m$ always a power of 2 for $m < n$?

155. If you answered Investigation **154** in the affirmative, find a formula which expresses $2^n \div 2^m$ as a single power of 2 whenever $m < n$.

156. Return to the definition of powers and show why the result in Investigation **??** really follows from the definition of powers. (I.e. provide a proof of Investigation **??**.)

157. Repeat Investigations **153-156** for a positive integer base different than $a = 2$.

158. Will the rules for exponents you found in Investigation **??** and Investigation **157** hold for any base $a > 0$? Explain why and how you know this.

159. The type of patterns and reasoning you used for $2^n \times 2^m$ and $2^n \div 2^m$ can naturally be extended to provide an analogous result for $(2^n)^m$ where m and n are positive integers. Find such a result and explain how you know it is valid.

The results you have found in Investigations Investigation **149**, Investigation **??**, and Investigation **159** are the classical "Rules for Exponents" that most of us were exposed to in Middle School.

160. Without wondering whether it is valid or not, apply your division rule from Investigation **??** to each of the expressions $2^5 \div 2^5, 2^3 \div 2^5$, and $2^4 \div 2^7$.

In Investigation **160** the division rule gave rise to an exponent which are 0 or even negative numbers. The definition of 2^4 should seem like second nature, we're used to "2 times 2 times 2 times 2." But 2^{-3}? You certainly "can't have a times itself a negative number of times."

Exponents and a suitable notation to express them is a human construct. It is part of a language - the language of algebra. Intuitive ideas and numerical patterns give rise to precise definitions. Yet this whole process would be of small value if the use of exponents in mathematics was limited to the narrow cases considered above.

Like any other language, mathematics grows to accommodate new needs. Here we look to extend the notion of exponents to include 0 and negative numbers. How do we do this? Patterns.

161. Complete the table that is begun below:

Row	Power Notation	Definition	Numerical Value
5	2^5	$2 \times 2 \times 2 \times 2 \times 2$	
4			16
3	2^3	$2 \times 2 \times 2$	
2		4	
1	2^1	2	

162. As you move from Row 1 of the table to Row 2 of the table, describe what happens to the entries in each column.

163. What happens to the entries in each of the columns as you move from Row 2 to Row 3? Row 3 to Row 4? Is there a pattern?

164. Now describe what happens to the entries in each column as you move from Row 5 to Row 4.

165. What happens to the entries in each of the columns as you move from Row 4 to Row 3? Row 3 to Row 2? Is there a pattern?

166. Following the pattern in Investigation **165**, you should be able to extend the table down another row - albeit leaving the definition column blank since we have no formal definition (yet) and no intuitive idea what should appear there.

167. Repeat Investigation **166** to extend the table to have five more rows, building meanings for the powers $2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}$, and 2^{-5}.

168. If they aren't already, convert the numerical values in each of the bottom five rows to fractions with numerators 1 and denominators a power of 2. Use this to provide a definition for powers with negative exponents:

Definition For bases $a > 0$ and exponents $m > 0$, define $a^{-m} = \frac{1}{?}$.

169. With this new definition, illustrate how your division rule for exponent works when $m \geq n$ as well.

Mathematicians have been able to extend our basic intuitive understanding of exponents to deal with essentially arbitrary bases and exponents. The most remarkabe illustration of how far we can extend the basic notion of exponents is certainly **Euler's formula**, discovered by **Leonhard Euler** (; -) around ????. The most elementary proof of the validity of this formula involves ideas from calculus relying on infinite series and periodic functions closely related to harmonics in music.

His formula is:

$$e^{i \cdot \pi} + 1 = 0.$$

Here π is the ubiquitous numerical constant related to circles and spheres, a transcendental irrational number which is approximated by the base-ten decimal $3.1415\ldots$ e is another fundamental mathematical constant, named after Euler himself, which arises in fundamental growth problems in biology, economics, and many other areas. It is also a transcendental irrational number which is approximated by the base-ten decimal $2.718\ldots$ i is the imaginary unit $i = \sqrt{-1}$ which gives rise to the complex numbers when combined with the real numbers. This number i is a square root we have long been told by our high school teachers does not exist despite the fact that "There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another." [6]

170. We've just noted that π, e, and i are fundamentally important numbers. What about 0 and 1? Are there other fundamentally important numbers that cannot be made from these five numbers? Explain.

171. What four fundamental mathematical operations are involved in Euler's formula? Are there fundamental operations that are not part of this formula? Explain.

172. In light of your answers to Investigation **170** and Investigation **171**, how remarkable is it that these five numbers and these four mathematical operations are expressed so concisely by this one formula? Explain, perhaps by attempting to create a simpler analogue or comparing with some other unifying statement from some other area of intellectual thought.

In regard to a similarly curious formula, $i^i = \frac{1}{\sqrt{e^\pi}}$, **Benjamin Pierce** (American Mathematician; 1809 - 1880), the "Father of American mathematics", said, "We have not the slightest idea of what this equation means, but we may be sure that it means something very important."

1.4. Concluding Reflections. **Albert Einstein** (; -) remarked, "It is not so very important for a person to learn facts. For that he does not really need a college. He can learn them from books. The value of an education in a liberal arts college is not the learning of many facts but the training of the mind to think something that cannot be learned from textbooks."

Essay 2: Compare and contrast the approach above for learning about exponents to that which you experienced in middle and/or high school. Relate this comparison/contrast to Einstein's quote, providing either supporting evidence for or dissenting views against Einstein's claim.

Noted mathematical author **Ian Stewart** (; -) once noted, "One of the biggest problems of mathematics is to explain to everyone else what it is all about. The technical trappings of the subject, its symbolism and formality, its baffling terminology, its apparent delight in lengthy calculations: these tend to obscure its real nature. A musician would be horrified if his art were to be summed up as 'a lot of tadpoles drawn on a row of lines'; but that's all that the untrained eye can see in a page of sheet music."

Essay 3: In this chapter we have described a few of the ways that common notions in mathematics are extended far beyond their original intuitive meaning. In this sense, is the language of the mathematician that much different in its history, development and accessibility much different than spoken languages? Much different from other formalized languages such as musical notation? Explain.

2. Teacher Notes

Investigation **130** is really a very subtle issue. At heart it speaks to the *completeness* of the real numbers which is actually part of their definition. [Is there an essay in here about .9999...?]

[6] Keith Devlin, from Mathematics: The New Golden Age.

CHAPTER 5

Patterns Linear and Arithmetic

In 1953 I realized that the straight line leads to the downfall of mankind. But the straight line has become an absolute tyranny. The straight line is something cowardly drawn with a rule, without thought or feeling; it is the line which does not exist in nature... Any design undertaken with the straight line will be stillborn. Today we are witnessing the triumph of rationalist knowhow and yet, at the same time, we find ourselves confronted with emptiness. An esthetic void, desert of uniformity, criminal sterility, loss of creative power. Even creativity is prefabricated. We have become impotent. We are no longer able to create. That is our real illiteracy.

Friedensreich Regentag Dunkelbunt Hundertwasser (Austrian Artist and Architect; 1928 - 2000)

The whole science of geometry may be said to owe its being to the exorbitant interest which the human mind takes in lines. We cut up space in every direction in order to manufacture them.

William James (American Psychologist; 1842 - 1910)

1. Introduction: The Line

Certainly there are significant limitations to a world populated only by the lowly line. Art would certainly be relatively crippled if it could employ only lines, limiting us to *line art* and *string art* like those pieces shown in Figures 1, 2, and 13. Perhaps that is what Hundertwasser meant in his lengthy quote above, or when he called the straight line "ungodly". Sections of Discovering the Art of Geometry in this series show us how *fractals* are one way mathematics has freed itself from it's most basic objects: lines, circles, and spheres.

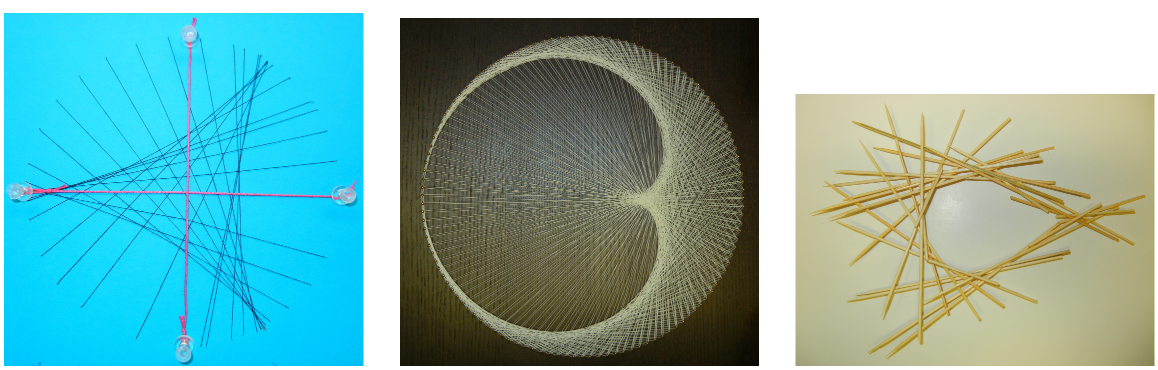


FIGURE 1. Original student string art.

Appreciating the freedoms that we can find in mathematics as well as art when we are freed from lines, we think Hundertwasser's condemnation is a bit too harsh. We believe there is significant merit in

William James' view - the line has natural resonances with the human mind. Throughout this chapter we provide short asides that illustrate the power and beauty of mathematical objects that are *linear* or *arithmetic*. We hope this removes some of the line's stigma.

FIGURE 2. Line Art

There is also a practical side to our approach. Students of mathematics as well as of art are often well-served by starting with a limited slate of objects to work with as they begin to explore these subjects. Once they have some familiarity with the general principles, ideas, and methods of these arts then the palets can be expanded fruitfully.

2. Chronophotography

[Stuff here about chronophotography. Horse issue. This time and motion stuff has important sociological connections (Frank and Lillian Gilbreth). Sports pysiology.]

We get a *sequence* of *equally timed* images. If we run through these images in *linear* time we get - a movie, television images, cartoons and animation, video! A basic way to see this is using flip books. [There were a number of precusors to video - ???give examples here. Yet all were based on a sequence of images being replaced linearly in time.]

The most important early work on chronophotgraphy was done by Marey and Muybridge. These "time and motion"studies are still fundamental to artists and animators - as well as many connections to other areas. Human locomotion was one of the first areas studied. To determine the movements of a human walking Marey dressed a man in a black velvet suit and had reflective *lines* along his upper spine, arm, and leg, as shown on the left in Figure **??**. The result is the striking image on the right in Figure **??**. This image and similar ones due to Maybridge were the impetus for Marcel Duchamp's Nude Descending Staircase (1912), one of the more important works of the early *Modernist* movement in art. The relationship is clear, as is the fundamental role that lines play.

3. Representing Information

How we display and represent data is fundamental to what impact it has on us. Edward Tufte's reknown books are a testament to this. Because we have different purposes at different times in mathematics it is no less important here.

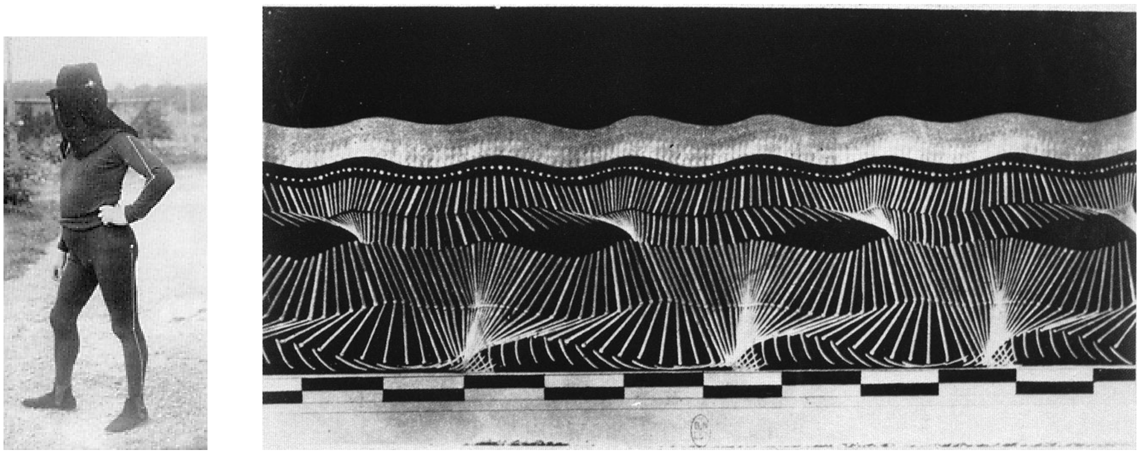


FIGURE 3. Man in black velvet and **tienne-Jules Marey** (French Scientist and Photographer; 1830 - 1904)

FIGURE 4. Nude Descending Staircase by **Marcel Duchamp** (French Artist; 1887 - 1968)

> To envision information - and what bright and splendid visions can result - is to work at the intersection of image, word, number, and art.[1]

[1] From the "Introduction" to Envisioning Information

Edward Tufte (American Statistician; 1942 -)

[Brief introduction into multiple representation as foreshadowing.]

4. A First Arithmetic Pattern - The Darbi-i Imam Frieze

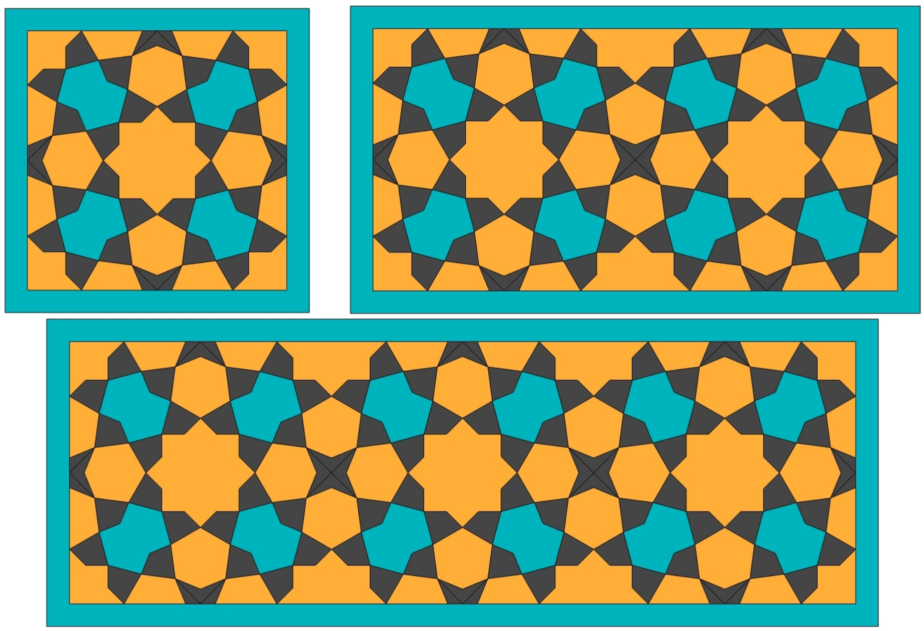


FIGURE 5. First three stages from a model of the *frieze pattern* on the Darbi-i Imam shrine.

In Section 3 we talked about different ways to represent data. Figure 5 are models which represent the first three stages in the construction of a tile ***frieze*** (a pattern which extends periodically in one direction). Made out of ceramic tiles like those which are found in mosaics throughout the world, the particular frieze being modeled is the entry portal of the Darbi-i Imam Shrine in Isfahan, Iran. The original frieze is pictured in Figure 9. These models provide ***physical representation*** of an underlying pattern.

1. If the frieze pattern in Figure 5 keeps growing in the *evident*[2] way, draw major features of the next three stages in this pattern.

Suppose that you were building this tilework frieze. It is of interest to know how many tiles of each type will be needed.

One ***numerical representation*** for the number of turquoise octagons required to complete the different stages of this pattern is the ***sequence***:

$$4, 8, 12, \ldots$$

Each entry in a sequence is called a ***term*** in the sequence.

2. Assuming the frieze pattern continues in the evident way, what are the next four terms in the turquoise octagon sequence?

[2]As noted in the Student Toolbox, any finite number of terms create infinitely many patterns. E.g. given the four terms $1, 3, 5, 7$, this pattern can be extended as $1, 3, 5, 7, 9, 11, 13, \ldots$ as odds, or as $1, 3, 5, 7, 11, 13, 17, \ldots$ as primes, or as $1, 3, 5, 7, 1, 3, 5, 7, 1, 3, 5, 7, \ldots$ just because, or ... So, when you see the word *evident* way it is just a caveat that we're hoping you might see the pattern the way we intended and not some unique way of your own. We'll continue to emphasize the term as a reminder.

The terms in a sequence come in a specific order and we specify their location using ***subscripts***; if we use the letter t to represent the turquoise sequence then t_1 represents the first term, t_2 the second term, etc. I.e. $t_1 = 4, t_2 = 8, t_3 = 12, \ldots$

3. What is the tenth term in the turquoise octagon sequence? I.e. $t_{10} =$?

4. What is the fiftieth term in the turquoise octagon sequence? I.e. $t_{50} =$?

Another numerical representation for the number of turquoise octagons that make up this pattern is as a ***table of values***:

n	t
1	4
2	8
3	12
⋮	⋮

5. Fill in the indicated values in the table for the turquoise octagon:

n	t
1	4
2	8
3	12
4	$\boxed{}$
5	
⋮	⋮
25	$\boxed{}$
⋮	⋮
75	$\boxed{}$
⋮	⋮
1000	$\boxed{}$
⋮	⋮
n	$\boxed{}$

6. By filling in the line representing the number of tiles needed in the n^{th} stage you have found an ***algebraic representation*** for the pattern. Write this representation as an algebraic formula: $t_n = \boxed{}$.

One way to look for patterns in numerical data is to compare successive terms in a sequence or table by considering their ***first differences*** as illustrated in Figure Figure 4. We use the capital Greek letter Δ to represent the first differences.

The growth of a numerical pattern is called **arithmetic** whenever the first differences in a sequence or table of values is constant.

Graphical representations are also important. There are many ways to represent our turquoise octagon data as a graph, as illustrated in Figure **??**.

Notice that the variables t and n continue to represent the number of turquoise octagons and the stage number respectively. The standard convention is to have the *independent variable* on the horizontal axis and the *dependent variable* on the vertical axis. Typically the Cartesian graph more regularly than the bar graph.

7. Extend the graph of the turquoise data to include all values of $n \leq 8$.

n	t		Δ
1	4		
		>	4
2	8		
		>	4
3	12		
		>	4
4	16		
⋮	⋮		⋮

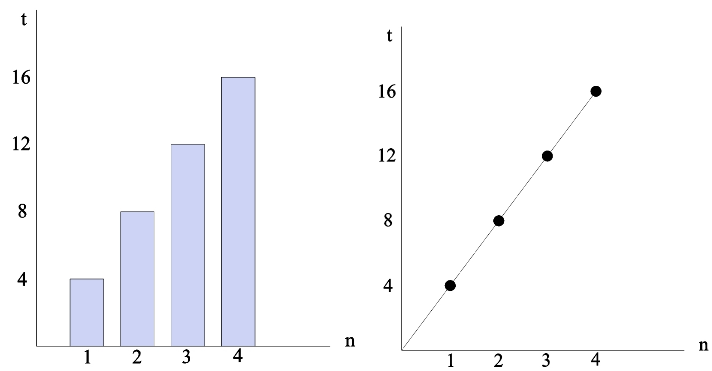


FIGURE 6. ***Cartesian graph*** and ***Bar graph*** of the turquoise octagon pattern.

8. The graph of the turquoise data in Figure 6 is called ***linear***. Explain why, providing a working definition of the term linear for use hereafter.
9. You should already know about the *slope* of a linear graph. Give a working definition of what the slope of a linear graph is. Then determine the slope of the graph in Figure 6.
10. You should know about the *vertical intercept* of a graph.[3] Give a working definition of what the vertical intercept of a graph is. Then determine the vertical intercept of the graph in Figure 6.

Functions of the form $f = m \cdot n + b$ where m and b are fixed numbers are called **linear functions**.

Now consider the beige hexagon tiles in the Darbi-i Imam frieze. Let's use the dependent variable b to represent the number of beige hexagons at each stage, counting only the number of whole hexagons that appear.

11. Represent the pattern of beige hexagons numerically as both a sequence and as a table of values. Provide six or eight terms of each.
12. Compute the first differences of both of the numerical representations in 11. Describe these first differences.
13. Is the pattern of beige hexagons in the frieze arithmetic?
14. Represent the pattern of beige hexagons in the frieze pattern graphically.
15. Is the graph in 14 linear? If not, how can it be described? If so, what are its slope and vertical intercept?
16. Represent the pattern of beige hexagons in the frieze pattern algebraically.
17. Is the function that describes the pattern of beige hexagons in the frieze pattern linear?

[3]When we denote the vertical axis by the dependent variable y the intercept is generally known as the y-intercept.

5. Paradigm Shift - The Darbi-i Imam Tessellation

In the study of ***crystals*** (e.g. diamonds, salt, ice, snowflakes, and quartz), five-fold symmetry was not seen and it was long thought that such symmetry could not exist in a natural crystal. This occurs because crystals that are created by nicely ordered structures all appear to have ***translational symmetry***, that is, they repeat periodically in a natural way.

Analogously, it was thought that in two-dimensional tilings, any collection of tiles that could *tessellate* could also be made to tessellate in a periodic way.

Mathematicians, physical scientists, artists, and craftspeople have thought about crystals and tilings throughout human history. So it was a paradigm shift when in the 1960's it became clear that ***aperiodic tiles***, tiles that would tessellate but could *never* be made to tessellate periodically, existed. In 1973 **Roger Penrose** (English physicist and mathematician; -) discovered a remarkably simple set of two tiles that were aperiodic. Shown in Figure 7, are Penrose's ***Kites and Darts*** which can be put together in many different ways to tessellate the plane, but cannot tessellate the plane in a periodic way.

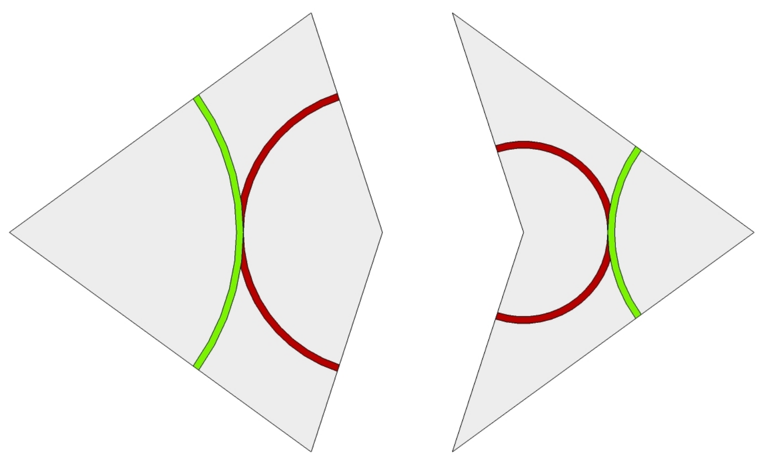


FIGURE 7. Penrose's kites and darts. (Note: Colors along edges must match for adjacent tiles.)

This celebrated discovery ushered in new urgency in the attempt to find three-dimensional analogues; naturally occurring ***quasicrystals***. They were found in 1982 by **Dan Shechtman** (Israeli physicist; -). Unfortunately, they were so revolutionary to accepted scientific doctrine that Shechtman was, in his own words, "a subject of ridicule and lectured about the basics of crystallography. The leader of the opposition to my findings was the two-time Nobel Laureate Linus Pauling, the idol of the American Chemical Society and one of the most famous scientists in the world." Pauling went so far to say, in reference to Shechtman, "There is no such thing as quasicrystals, only quasi-scientists." Shechtman perservered, saying, "For years, 'til his last day, he [Pauling] fought against quasi-periodicity in crystals. He was wrong, and after a while, I enjoyed every moment of this scientific battle, knowing that he was wrong."[4]

Shechtman's perserverence paid off. On 5 October, 2011, he was awarded the Nobel Prize in Chemistry "for the discovery of quasi-crystals". The Press Release by the Nobel Committee is particularly interesting:

[4]From "Ridiculed crystal work wins Nobel for Israeli" by Patrick Lannin and Veronica Ek, Reuters, 8/5/2011.

> **A remarkable mosaic of atoms**
>
> In quasicrystals, we find the fascinating mosaics of the Arabic world reproduced at the level of atoms: regular patterns that never repeat themselves. However, the configuration found in quasicrystals was considered impossible, and Dan Shechtman had to fight a fierce battle against established science. The Nobel Prize in Chemistry 2011 has fundamentally altered how chemists conceive of solid matter.
>
> On the morning of 8 April 1982, an image counter to the laws of nature appeared in Dan Shechtman's electron microscope. In all solid matter, atoms were believed to be packed inside crystals in symmetrical patterns that were repeated periodically over and over again. For scientists, this repetition was required in order to obtain a crystal.
>
> Shechtman's image, however, showed that the atoms in his crystal were packed in a pattern that could not be repeated. Such a pattern was considered just as impossible as creating a football using only six-cornered polygons, when a sphere needs both five- and six-cornered polygons. His discovery was extremely controversial. In the course of defending his findings, he was asked to leave his research group. However, his battle eventually forced scientists to reconsider their conception of the very nature of matter.
>
> Aperiodic mosaics, such as those found in the medieval Islamic mosaics of the Alhambra Palace in Spain and the Darb-i Imam Shrine in Iran, have helped scientists understand what quasicrystals look like at the atomic level. In those mosaics, as in quasicrystals, the patterns are regular - they follow mathematical rules - but they never repeat themselves.
>
> When scientists describe Shechtman's quasicrystals, they use a concept that comes from mathematics and art: the *golden ratio*. This number had already caught the interest of mathematicians in Ancient Greece, as it often appeared in geometry. In quasicrystals, for instance, the ratio of various distances between atoms is related to the golden mean.
>
> Following Shechtman's discovery, scientists have produced other kinds of quasicrystals in the lab and discovered naturally occurring quasicrystals in mineral samples from a Russian river. A Swedish company has also found quasicrystals in a certain form of steel, where the crystals reinforce the material like armor. Scientists are currently experimenting with using quasicrystals in different products such as frying pans and diesel engines.

You may be surprised to see mention of the Darbi-i Imam shrine in this citation, aperiodic tilings and quasi-crystals as we have described them are recent discoveries.

Once again, as with the geocentric model of the solar systems and a non-flat earth, this is largely a function of our self-importance and our lack of respect for the brilliance of the ancient scholars, artists and craftspeople.

In 2005, while visiting the Middle East, graduate student **Peter Lu** (; -) became very interested in the tile-work on the Darbi-i Imam shrine. When he returned to Harvard he set to work studying these tilings. What he discovered was shocking:

> The asymptotic ratio of hexagons to bowties approaches the golden ratio τ (the same ratio as kits to darts in a Penrose tiling), an irrational ratio that shows explicitly that the pattern is quasi-periodic. Moreover, the Darbi-i Imam tile pattern can be mapped directly into Penrose tiles.[5]

Iranian craftspeople had predated the discoveries of Penrose by over 500 years!

[5] From "Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture" by Peter J. Lu and Paul J. Steinhardt, *Science*, Vol. 314, 23 February, 2007, pp. 1106-1110.

The (combined) mathematical and artistic study of medieval tilings such as these is rich and beautiful.[6]

FIGURE 8. Entry portal from Dabri-i Imam in Isfahan, Iran. Photo courtesy of Peter Lu.

FIGURE 9. Darbi-i Imam frieze detail.

18. What are some other examples in science and/or mathematics where there were particularly personal attacks/disagreements between scientists?

19. What would you have done had you found yourself in Shechtman's place in this controversy?

[6]One fun place to start is building with these tiles. See `http://www.3dvinci.net/mathforum/GirihTiles_StudentVersion.pdf` where Google SketchUp is used to help create tilings with the tiles that are found in the Darbi-i Imam.

FIGURE 10. Peter Lu.

6. Multiple Representations of Patterns

The frieze patterns above appeared as physical patterns. But we can just as well start with a function represented algebraically. For example,

$$f = 3 \cdot n + 7.$$

20. Is the function f linear?
21. Represent the function f numerically as both a sequence and as a table of values. Provide six or eight terms of each.
22. Compute the first differences of both of the numerical representations in Investigation **21**. Describe these first differences.
23. Is the numerical data formed by f an arithmetic pattern?
24. Represent the function f graphically.
25. Is the graph in 24 linear? If not, how can it be described? If so, what are its slope and vertical intercept?
26. What about physically? Suppose you worked with tiles. Is there a natural way to show how we can represent the function f physically as a growing pattern of tiles?

Now consider new data, which is given graphically as in Figure 11. Assume that this graph continues in the *evident* way.

27. Is the graph in 11 linear? If not, how can it be described? If so, what are its slope and vertical intercept?
28. Represent the function g numerically as both a sequence and as a table of values. Provide six or eight terms of each.
29. Compute the first differences of both of the numerical representations in 28. Describe these first differences.
30. Is the numerical data formed by g an arithmetic pattern?
31. Represent the function g algebraically.
32. Is the function g linear?
33. Represent the function g physically.

Consider the sequence s given by $7, 11, 15, 19, \ldots$

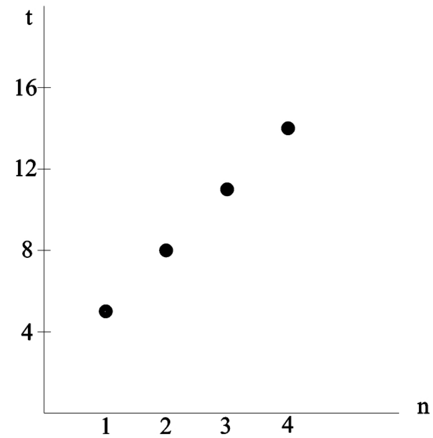


FIGURE 11. Graph of the data from a function g.

34. Represent the function s numerically as a table of values. Provide six or eight terms.
35. Compute the first differences of both of the numerical representations in 34. Describe these first differences.
36. Is the numerical data formed by s an arithmetic pattern?
37. Represent the function s graphically.
38. Is the graph in 37 linear? If not, how can it be described? If so, what are its slope and vertical intercept?
39. Can you find a way to represent the function s physically.
40. Represent the function s algebraically.
41. Is the function s linear?

Now it's time to make your own arithmetic pattern as a growing frieze. Figure 12 below shows a growing arithmetic pattern that is constructed using ***pattern blocks***, a collection of six different, brightly colored tile blocks that are often used in elementary school mathematics classrooms.

42. Use pattern blocks or one of the online pattern block applets (e.g. `http://nlvm.usu.edu/en/nav/frames_asid_170_g_2_t_3.html`) to build a growing frieze pattern where at least one of the types of blocks illustrates arithmetic growth.
43. Graph your arithmetic pattern.
44. Find the algebraic representation of your pattern.

7. Meta Patterns

We have been considering single functions/patterns to see if they were linear/arithmetic. Now we would like to see if there are patterns that unite what we have learned about these patterns. Such a pattern could be called a *meta pattern*.

45. If you have a linear function $f = m \cdot n + b$ what can you say about its graphical representation? Its numerical representations? Its physical representation? Explain.

46. If you have a function whose graph is linear, what can you say about its algebraic representation? Its numerical representation? Its physical representation? Explain

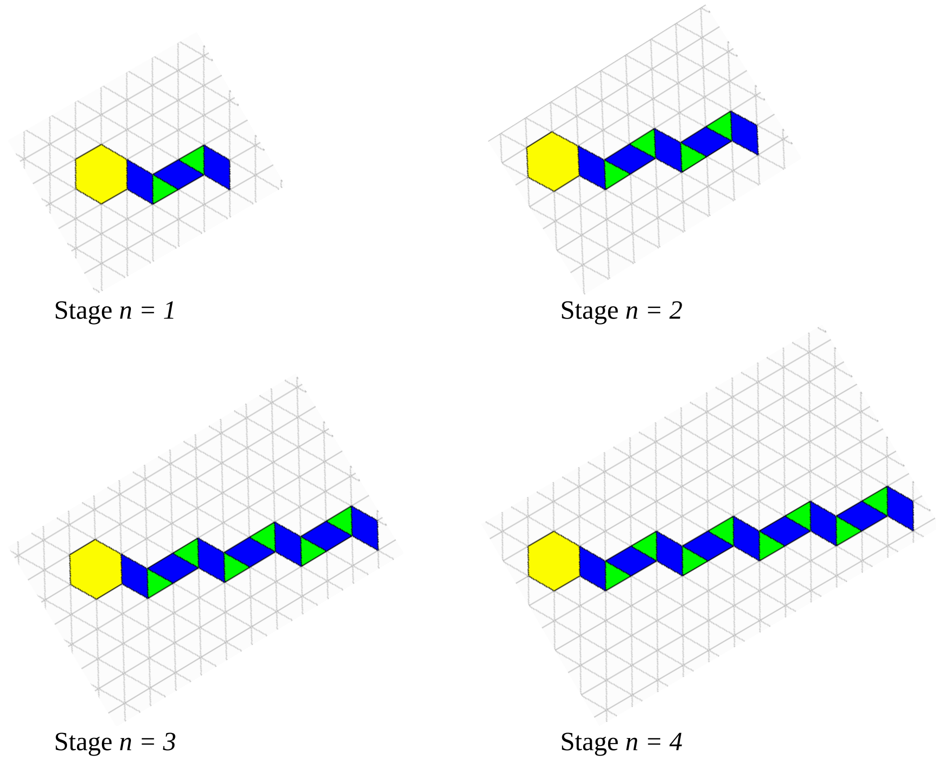


FIGURE 12. Growing caterpillar pattern constructed from pattern blocks.

47. If you have a function whose numerical data is arithmetic, what can you say about its algebraic representation? Its graphical representation? Its physical representation? Explain.

8. Connections

[Some of these maybe should be included above to break up the other sections.]

8.1. Pattern Block Patterns. It is difficult to overemphasize the power of simple manipulatives like pattern blocks to nurture the creative spirit. By all means, try to find the opportunity to create your own mosaic.

For those interested in teacher, there are many wonderful resources which describe or model the use of such manipulatives in elementary teaching. For examples, "Case 19: Growing Worms 1" and "Case 20: Growing Worms 2" in Discovering Mathematical Ideas: Patterns, Functions and Change Casebook by Deborah Schifte, Virginia Bastable and Susan Joe Russell.

8.2. Linear Programming.

> The linear-programming was – and is – perhaps the single most important real-life problem.[7]
>
> **Keith Devlin** (; -)

Linear Programming Not sure where this is going to go in general, but there certainly needs to be a big hook here.

Put in a simple example? The lemonade example from way back in the day?

8.3. Origami. Origami EVERY one of these shapes is made by folding lines!!! There is nothing else.

8.4. Shadows. Light/CAT scans/Shadows All of these things are just the lines made by light. CAT scans are just lines and look what they tell us about our 3D bodies!!

8.5. Linear Regression. Linear Regression A fundamental application. That it is applied so much gives us some sense of how many things exhibit approximately linear growth.

8.6. Art. Perspective Drawing Give links to the appropriate sections in the Geometry book.

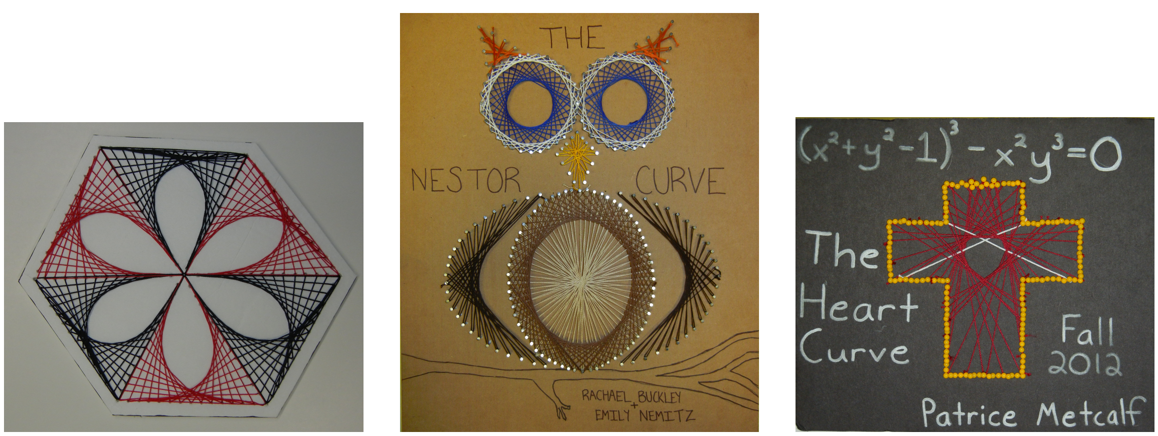


FIGURE 13. Original student string art.

8.7. Calculus. Rates and Calculus Anything that is a rate is linear by implication. In other words, lines are what tell us all about Calculus. So there needs to be big hooks to this book.

[7]From Mathematics: The New Golden Age, p. 605.

Points. Have no parts or joints. How can they combine. To form a line?

J.A. Lindon (; -)

CHAPTER 6

Patterns and Problem Solving

The tantalizing and compelling pursuit of mathematical problems offers mental absorption, peace of mind amid endless challenges, repose in activity, battle without conflict, refuge from the goading urgency of contingent happenings, and the sort of beauty changeless mountains present to senses tried by the present-day kaleidoscope of events.

Morris Kline (American Mathematician; ??? - 1992)

It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts.

Paul Halmos (??? Mathematician; ??? - ???)

Almost every American who has a degree, however ignorant, can live better than even competent people in much poorer countries around the world... But this cannot last long in the situation when "competence" and a diploma tautologically mean each other. The advantages enjoyed by Americans are the results of real competence and real efforts of previous generations... And someday ignorant people with degrees and diplomas may want power according to their papers rather than real competence. We Russians have some experience of this sort... It is clear to me right now that the winners in the modern world will be those countries which will really teach their students to think and solve problems. I sincerely wish America to be among these.

Andre Toom (Russian Mathematician; 1942 - ???)

The problem is not that there are problems. The problem is expecting otherwise and thinking that having problems is a problem.

Theodore Rubin (American Psychiatrist; 1923 - ???)

The best way to escape from a problem is to solve it.

Alan Saporta (???; ??? - ???)

We only think when confronted by a problem.

John Dewey (American Educator; 1859 - 1952)

When I am working on a problem, I never think about beauty. I only think of how to solve the problem. But when I am finished, if the solution is not beautiful, I know it is wrong.

Buckminster Fuller (American Architect; 1895 - 1983)

1. Introduction

We all face problems each and every day. What an amazing thing that brains have developed to help us solve some of these problems. It is a miraculous thing. Give stories about amazing things that animals can do to help them solve problems, showing sorts of higher order thinking skills.

We take such "routine" problem solving as routine. But it is not. Think of all of the problems that have been solved to make life as "simple" as it is. 5,000 years ago there was no metal. Now for a few days wages you can afford to wear your iPod onto an airplane that will fly at 50,000' at 600 miles per hour and take you to New York City, a city where over 8 million people live in an area of just over 300 square miles. The problem solving that has enable this to happen is immense.

(COOL!! Historical, philosophical, and intellectual context!)

Because solving problems is so important, humankind has thought a great deal about how we solve problems. Meno, etc. Links to the philosophy stuff here.

Socrates/Plato

Meno?? (See p. 99 of Hersch; knowledge is innate. Links to our pedagogical style.)

Descartes Discourse on Method, Optics, Geometry, and Meterology. His four simple rules were philosophical analogues of the revolutionary views that were fanning the Protestant revolution in parallel. Philosophy tied with deep mathematics and optics. Great quote by D'Alembert on p. 111 of Hersch.

The rainbow is such a remarkable phenomenon of nature, and its cause has been so meticulously sought after by inquiring minds throughout the ages, that I could not choose a more appropriate subject for demonstrating how, with the method I am using, we can arrive at knowledge not possessed at all by those whose writings are available to us. Rene Descartes, attributed in The Rainbow Bridge (p. 182) to De l'arc-en-ciel from Discours de la methode. Wonderful that this was part of the Discourse.

Bacon - Scientific Revolution

Polya - How to Solve It.

Kuhn - Structure of Scientific Revolutions

Lakatos - Proofs and Refutations.

Must include links to deeper questions about the way you reason limiting what you can reason about. This latter stuff deserves more complete airing in the Proof book. This is a great way to delineate the split here. Write much of this as one piece. Then dole it out in smaller measure more tailored to problem solving for the Patterns book and it broader measure more philosophically/epistemologically in Proof book.

Prominent Philosophers Links to Mathematics

Husserl (phenomenology) wrote his Ph.D. on the Calculus of Variations under Weierstrass; Bertrand Russell; Descartes - Cogito and Cartesian Geometry; Newton - A philopher? Leibniz - How to describe this in an accessible way? Frege, Hilbert, and Godel - Do many philosophers or philosophy students even read about them? Kant - Much of his discussion of synthetic a priori is geometrical. (Links to Meno?) Plato; Pythagoreans; C.S. Pierce - Father of Pragmatism; important mathematician; Wittgenstein, Godel and the Vienna Circle; Lakatos first worked with Renyi in 1954 at the Hungarian Mathematical Research Institute as a translator. One of these books was Polya's How to Solve It. In 1958 he met Polya who later becam his dissertation advisor and thus Proofs and Refutations was born. Spent more time on Philosophy of Science afterward, including arguing with Kuhn. (See Hersch, pp. 208-216.)

So maybe you don't want to be somebody who thinks about how we solve problems: a philosopher, a psychiatrist, or an educational theorist. About now you might be bringing out the oft-used contemporary mantra, "When will I need to know this?" Show me the money! We hope that your work through the material in these guides will help you develop an understanding of why this is an unfortunate and limiting attitude. Whether you have arrived at this point yet, will arrive there, or even differ with us - regarding learning in general and mathematics in particular - nobody can deny that they will always need to know how to solve problems. Each day brings forth a wealth of problems to solve. And there are many benefits to solving problems:

POLYA!!!!!!!!!!!!!!!!

> The value of a problem is not so much coming up with the answer as in the ideas and attempted ideas it forces on the would be solver.
>
> **I.N. Herstein** (??? Mathematician; ??? - ???)

> Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.

George Polya (??? Mathematician; ??? - ???)

> Solving problems is a practical art, like swimming, or skiing, or playing the piano if you wish to become a problem solver you have to solve problems.

George Polya (; -)

> The problems that exist in the world today cannot be solved by the level of thinking that created them.

Albert Einstein (German Physicist; 1879 - 1955)

2. Three Motivating Problems

Our work in this lesson will be motivated by three problems:

The Handshake Problem: A number of people are to be introduced to each other by shaking hands. If each person shakes hands with every person (excluding themselves) exactly once, what is the total number of handshakes that must be are made?

The Circle Problems: On a circle n points are drawn. Lines are drawn which connect each of the points to all of the other points on the circle. Circle Problem 1: How many lines must be drawn to connect all of the points? Circle Problem 2: Into how many regions is the circle decomposed by the lines?

Example:

Four points yielding six lines and eight regions.

???Image here

The Line Problem: On a plane n lines are drawn. Each line intersects every line exactly once and no more than two lines intersect at a single point. Line Problem 1: How many points of intersection are formed by the lines? Line Problem 2: Into how many regions is the plane divided by the lines?

Example:

Four lines yielding six points of intersection and eleven regions.

???Image here

These are your problems to solve. You should spend significant time working on these problems before you move on. As wrestle with these problems, be reminded of what John Dewey told us:

> No thought, no idea, can possibly be conveyed as an idea from one person to another. When it is told it is to the one to whom it is told another fact, not an idea... Only by wrestling with the conditions of the problem at first hand, seeking and finding his own way out, does he think.

Indeed, the whole purpose of this series is to get you working on problems, to get you thinking and for you to be mathematically active. Here the problems are both clear enough and meaty enough that we can give them to you without much guidance. We are confident that you can make significant headway on these problems on your own.

3. Problem Solving Strategies

Here we provide some strategies that can be used to help solve ??? and related problems. These strategies are presented in the same guided discovery framework that typical investigations are. In how much detail you decide to consider these sections depends on your success in finding patterns in the original problems, your success in completing the investigations, and the directions/requirements of your teacher.

Did you spend an hour working on the problems above? Have you solved them to your satisfaction? If not, go back and do this. This chapter will not be successful if you have not done this. And then what? It depends on you, your teacher, and the nature of your solutions to these problems. Our hope is that in your investigation of these problems you have found sufficiently robust ideas, strategies, relationships, and patterns that you can solve the problems in the penultimate section "Using what you have discovered/learned." For now skip ahead to this section and try out some of these investigations. Yes, we said, SKIP AHEAD TO THE END. You'll know when and if you need to come back

So you're back. Yeah, we expected you might be. There is a reason for the topics and investigations in between. Your idea of "solving" these problems may not have been robust enough to help you solve all of the investigations at the end. Your strategies might be limited, allowing you to solve only certain problems in the final section. Etc. The intervening sections provide guided prompts for a variety of different strategies that are both typical and effective means of solving the three problems you've been working on. Maybe you need some helping solving one or more of the three problems - work through a section that may seem to be related to your strategy if you are stuck. Try one of these if you need a push. Work through some of these sections if your strategies did not apply to some of the investigations at the end and you need a new approach. Or just work through these intervening sections to see what you can discover there. It's up to you. (And perhaps your teacher.) Additionally, there are many ways that you can choose to work on these problems cooperatively with peers, including the wonderful Jigsaw Classroom method. ???Footnote here.

We learned all of these things from students!!!!!!!!!!! These are not ideas handed down, they did it. Their diversity reflects the diversity of our experiences, our learning styles, and our ideas.

Big Idea - The diversity of ways that you can solve a single problem. Moreover, how very much you can learn if you look at things from a different perspective. Some strategies give rise to whole classes of problems that can be readily solved by these means while not being useful to other whole classes of problems.

Teacher notes: The premise of the Problem Solving Strategies section is that students will have many of their own ideas and strategies. But these subsections should demonstrate that there is a great deal of important mathematics that can be accessed via these problems. As a teacher it will be your job to adjust pacing, emphasis, assignments, classroom dynamics, and the like. Here are some suggestions for you: Jigsaw Classroom Assign specific sections of interest, leave the rest to students as necessary. Extensive work only on the problems at the outset and leave this sections to students as necessary. Assessment on some sections but not others. Assessment only on the final section.

3.1. Problem Solving Strategy: Collecting Data. In each of the ??? problems above there is a variable - the number of people, points, and lines - which is indeterminate. You need to solve the problem no matter how many of each they are. A natural response to this is to collect some data and see what this quantitative data might tell you.

Let's do that here.

Below are circles with two, three and four points on the circle. These points are connected by straight lines as shown.

With n, P, R, and L, representing the stage number, number of points, number of regions, and number of lines, the properties of the figures can be tabulated as follows:

n 1 2 3 4 5 P 2 3 4 L 1 3 R 2 4

48. Determine the values of R and L for the 3rd stage.
49. Assuming the pattern continues in the indicated way, draw the 4th stage in this pattern.
50. 3. Use 2) to determine the values of P, R, and L for the 4th stage.
51. 4. Now draw the 5th stage in the pattern.
52. 5) Use 4) to determine the values of P, R, and L for the 5th stage.
53. 6) Make a conjecture which describes how we can predict the value of P as an explicit function of the stage number n.
54. 7) Make a conjecture which describes how we can predict the value of R as an explicit function of the stage number n.

55. 8) Find and describe a pattern in the values of L as a function of the stage number n. Is it easy to find an explicit function which describes L as a function of n?

Now let's switch to the lines problem.

56. 9) Draw two lines, extending indefinitely, that are neither concurrent nor parallel.
57. 10) How many points of intersection are there in the pair of lines you drew in 9)?
58. 11) Draw three lines, extending indefinitely, so that no pair are concurrent, no pair are parallel, and only two lines cross at any point of intersection.
59. 12) How many points of intersection (where at least two lines meet) are there among the three lines you drew in 11)?
60. 13) Repeat 11) for four lines.
61. 14) How many points of intersection (where at least two lines meet) are there among the four lines you drew in 13)?
62. 15) There is a pattern in your answers to 10), 12), and 14). Describe it.

16) Use 15) to make a conjecture about the number of points of intersection if you draw five lines, six lines, and seven lines.

Now let's try the handshake problem where you might want to have a small group of people nearby to test things out.

How many handshakes will there be when there are

17) two people in the room?
18) three people in the room?
19) four people in the room?
20) ... five people in the room?

21) Complete the table of values below. You should see a pattern forming. Describe the pattern in detail.

n	h
2	
3	
4	
5	
6	
7	

22) How is this problem related to the problems considered in Sections I and II? Explain in detail. In particular, if you were to make tables for the data of these earlier problems, how would they compare?

3.2. Problem Solving Strategy: Gauss's Epiphany. Carl Friedrich Gauss German Mathematician (1777; 1855 - i)s, without doubt, one of the greatest mathematicians that ever lived. His work is explored in detail in <u>Discovering the Art of Number Theory</u> in this series. This strategy and group of explorations is named after an epiphany widely attributed to him:

From The Joy of Mathematics: Discovering Mathematics All Around You by Theoni Pappas, Wide World Publishing/Tetra, 1986.)

23) Use this method to determine how many handshakes there are if there are 27 people in a room.

24) Use this method to determine how many points of intersection there are if 53 lines in the plane are drawn according to the approach in 3.2.

25) Use this method to determine how many lines are needed to connect 85 points around a circle according to the approach in Section 3.1.

Each of the problems 23) - 26) involve determining the value of an arithmetic series of the form:

$$1 + 2 + 3 + \ + (n-1) + n.$$

26) Use Gauss's method to determine a formula for the series $1+2+3+\ +(n-1)+n$ as a function of the variable n.

27) Check that the formula for in 26) provides the correct answers for investigations 23) - 26).

28) Each of the investigations in 23) - 26) involved odd numbers so the value of n used in 27) are all even. What potential difficulty is there when n is odd? Resolve this difficulty and precisely describe your resolution.

29) What is the sum of the series

$$1+2+3+\ +1,345,217+1,345,218+1,345,219?$$

30) What do you think of this method?

3.3. Problem Solving Strategy: Blocks and Other Manipulatives. In the illustration below, a student determines the value of $1+2+3+4+5+6$ using Multilink cubes - plastic cubes that join together. By creating two identical staircases out of $1+2+3+4+5+6$ blocks each and joining them together to form a 7 by 6 rectangle, she shows that $1+2+3+4+5+6 = 42/2 = 21$. [1]

31) Use this method to determine how many handshakes there are if there are 27 people in a room.

32) Use this method to determine how many points of intersection there are if 53 lines in the plane are drawn according to the approach in Section 3.2.

33) Use this method to determine how many lines are needed to connect 85 points around a circle according to the approach in Section 3.1.

Each of the problems 31) - 33) involve determining the value of an arithmetic series of the form:

$$1+2+3+\ +(n-1)+n.$$

34) Use Gauss's method to determine a formula for the series $1+2+3+\ +(n-1)+n$ as a function of the variable n.

35) Check that the formula for in 34) provides the correct answers for investigations 31) - 33).

36) What is the sum of the series

$$1+2+3+\ +1,345,217+1,345,218+1,345,219?$$

37) What do you think of this method?

3.4. Problem Solving Strategy: Combinatorics - The Art of Counting. The important online mathematical encyclopedia mathworld.wolfram.com defines combinatorics as follows: Combinatorics is the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties. More colloquially, combinatorics is the mathematical art and science of counting. It is a critical tool in many areas of mathematics and relies heavily on patterns. Hmmm..., it must be exactly what we need to solve the ??? handshake problem since it is all about counting.

Here's an example of combinatorial reasoning. A town sports league has each team play every other team exactly twice, once as the home team and once as the visiting team. How many games must they schedule? With 2 teams there are clearly only 2 games. With 3 teams you can check that there are 6 games. And with 4 teams there are 12 games. What if there were 24 teams? It seems complicated. But we can reason as follows. As a home team, each team must play 23 games, one with each of the teams in the league. Since this is true for each team and there are 24 teams, there are $23\ \ 24 = 552$ games.

38) Describe precisely what's wrong with the following argument for determining the number of handshakes with 12 people in the room: Each person must shake hands with 11 other people. Since there are 12 people, there are $12\ \ 11 = 132$ handshakes.

[1]From Essentials of Mathematics: Introduction to Theory, Proof, and the Professional Culture by Margie Hale, Mathematical Association of America, 2003.

39) Despite it's incorrectness, the method in 38) can be adapted to provide the appropriate result. Explain.

40) Use 39) to determine the number of handshakes if there are 27 people in a room.

41) Use 39) to determine the number of handshakes if there are 532 people in a room.

42) Use 40) to determine to determine a formula for the number of handshakes as a function of the number n of people in a room.

43) Suppose that 53 lines in the plane are drawn according to the approach in Section II. Translate this problem into a handshake problem and determine the total number of points of intersection.

44) Suppose 85 points around a circle are connected with lines according to the approach in Section I. Translate this problem into a handshake problem and determine the total number of lines.

45) What do you think of this method?

3.5. Problem Solving Strategy: Discrete Calculus. In Topic ?? we saw that the first differences of the terms in a numerical sequence are constant precisely when the sequence is generated by a linear equation of the form (i.e. a function of the form $f = mn + b$.) In Discovering the Art of Calculus in this series we extend this result to a much more general pattern at the heart of a discrete calculus:
Fundamental Theorem of Discrete Calculus Let S be a sequence. The sequence of k^{th} differences is the lowest degree of constant first differences if and only if the original sequence is generated by a k^{th} degree polynomial function.

We can use this profitably to solve problems like those considered in Sections I - III.

46) Copy the table of data from Investigation 21), denoting the dependent data by f instead of h. Fill in the first differences.

47) Does this data belong to a linear equation? Explain.

48) On the table in 46) fill in the second differences.

49) What does 48) tell you about the type of function that generates the handshake data?

You should remember the quadratic formula from high school algebra. It is used to calculate the roots of the general quadratic function . This function is also known as the general 2nd degree polynomial function, which you can now use. You must only determine the values of a, b, and c.

50) Use results from earlier sections to determine appropriate values of a, b, and c.

Here's similar data that also has constant second differences:

n	f		1st Δ		2nd Δ
0	2				
		>	4		
1	6			>	2
		>	6		
2	12			>	2
		>	8		
3	20			>	2
		>	10		
4	30			>	2
		>	12		
5	42			>	2
		>	14		
6	56				

This data must also be described by a quadratic function .

51) Substitute $n = 0$ into the quadratic and solve for c.

52) Substitute $n = 1$ into the quadratic to generate an equation containing only the variables a and b.

53) Repeat 52) with $n = 2$.

54) Solve the equations in 52) and 53) simultaneously to determine the values of a and b.

55) Write out the quadratic explicitly and show that it correctly generates the appropriate values for $n = 0, 1, 2, , 6$.

56) What do you think of this method?

3.6. Problem Solving Strategy: Factoring. Often we cannot see patterns simply because there are several intertwined processes at work that can only be understood once they are isolated. For number sequences, factoring can often help us untangle these processes so we can describe the underlying processes that give rise to the pattern.

57) Let's return to the pattern in Section VI:

n f Factors of f 0 2 1 6 2 12 112 or 26 or 34 3 20 4 30 5 42 6 56

Fill in the remaining factors of f that have not been filled in.

58) Choosing appropriate factors from each row, there is a very regular pattern to the factors. Highlight these factors and describe the pattern precisely.

59) Describe the pattern of smaller factors from 58) as a function of n.

60) Describe the pattern of larger factors from 58) as a function of n.

61) Suitably combine 59) and 60) to determine an explicit formula for the pattern f as a function of n. Check that this function provides the appropriate data for $n = 0, 1, 2, , 6$.

We'd like to use this approach for the handshake problem as well.

62) Use 21) to fill in the table below and then determine factors of the data for f in the middle column: n h Factors of h 2 3 4 23 5 6 7

63) You should see a clear pattern among some subset of the factors. Describe it precisely.

The pattern in 63) is not easy to describe directly. It is easier if we force a factor of into each term. In other words, instead of 6 = 23 we write . 64) Rewrite the table in 62) forcing a factor of into each of the factors you have. Precisely describe the pattern in the factors that you see.

65) Following the examples of 59) - 61), determine an explicit formula for h as a function of n.

66) Compare your expression for h in 65) with the results of previous sections.

3.7. Problem Solving Strategy: Algebra. Note that this builds on the Factoring section.

3.8. Problem Solving Strategy: Recursion. One meaning of the word recur is to happen again. While we have not defined patterns, there is something inherent in our understanding of this term that in a pattern something happens again, and again, and again,... Mathematicians have adopted this root and use the word **recursion** to name a process in which objects are defined relative to prior objects in the same process. They also use the term ***recursive*** as the associated adjective.

While you may not have heard the term before, the importance of recursion in the world around us cannot be understated. Populations, weather, account balances, and many other real phenomena we might study are dependent at any stage on their size, behavior, distribution, and makeup at prior stages. Anybody who has used a spreadsheet has used recursion when they define a new cell using information in other cells. Recursion underlies the develoment of fractal and chaos described in Discovering the Art of Geometry.

In an important sense, to discover a recursive relationship is to really see what it is that makes the object being studied a pattern. They can also be helpful in problem solving.

At each stage, add... Do a table with first differences shown. Do it in English. Do it using recursive notation. Multiple representations!!! It is done physically already. Factorial as a choosing problem. Links with the fundamental counting principle. Need to say something about how one can solve recursive relationships. This is huge and there is an entire branch of mathematics devoted to this. ???Some ILL books are coming on this. Find a good linear growth pattern among Frieze patterns in a famous place.

???Need a note that the triangular numbers are the additive analogue of factorials. Not sure if this needs to be here or elsewhere.

???Teachers Notes - Note about the most basic recurrence relations subsuming a tremendous amount of mathematics. $r_n = n \cdot r_{n-1}$ gives rise to factorial,$r_n = n + r_{n-1}$ gives rise to triangular numbers, $r_n = k \cdot r_{n-1}$ gives rise to exponential growth, and $r_n = k + r_{n-1}$ to arithmetic growth.

63. A joke among mathematicians is that a dictionary had the following definition:

recursion \ ri-'k ər-zh ən\ n See "recursion".

Explain this joke. To be funny jokes generally have some kernel of truth in them. What underlying truth does this joke point out about languages and/or dictionaries?

In 3.5 we represented the data in our pattern numerically in a table and considered the associated differences. In this case we have

n	f		1st Δ
1	8		
		>	5
2	13		
		>	5
3	18		
		>	5
4	23		
		>	5
5	28		
		>	5
6	33		

4. Trying Out What You Have Discovered

Now comes your chance to utilize the strategies you have discovered and/or learned. You should be able to answer each of the questions in this section. If not, you should go back and work to find a method that you can use. Also, please note that most problems can be solved using several different methods.

100) How many handshakes are there if there are 27 people in a room?

101) How many handshakes are there if there are 532 people in a room?

102) Determine a closed-term, algebraic expression for the number of handshakes (represented by the dependent variable h) there are if there are n people in a room.

103) Determine a recursive relationship for the number of handshakes when there are n people in the room (represented functionally by) as a function of the numbers of handshakes with fewer people in the room (denoted by respectively).

104) How many points of intersection are there if 53 lines as described in the Line Problem?

105) How many points of intersection are there if 264 lines as described in the Line Problem?

106) Determine a closed-term, algebraic expression for the number of points of intersection (represented by the dependent variable p) there are if there are n lines drawn as described in the Line Problem.

107) Determine a recursive relationship for the number of points of intersection (represented functionally by) as a function of the number of points of intersection when there are fewer lines (denoted by respectively).

108) How many lines are needed to connect 85 points around a circle as described in the Circle Problem?

109) How many lines are needed to connection 388 points around a circle as described in the Circle Problem?

110) Determine a closed-term, algebraic expression for the number of lines needed (represented by the dependent variable l) if there are n points connected as described in the Circle Problem.

111) Determine a recursive relationship for the number of lines needed (represented functionally by) as a function of the numbers of lines needed with fewer points around a circle (denoted by respectively).

112) Determine the sum of the series 5 + 10 + 15 + 20 + + 7,895.

113) Determine the value of the sum of the following series: 1 + 3 + 5 ++ 2101 + 2103 + 2105.

114) Determine the value of the sum of the following series: 1 + 4 + 7 ++ 158509 + 158512 + 158515.

115) Determine the value of the sum of the following series: 8 + 10 + 12 ++ 11212 + 11214 + 11216.

116) Determine the value of the sum of the following series: 284 + 291 +298 + 305 + + 3056 + 3063 + 3070.

117) Each of the series in 112) - 116) are called arithmetic series. In general, such a series can be written as: . Determine a closed-term, algebraic function for the sum of this series as a function of the parameters a, d, and n.

118) Use the formula in 117) to check your answers to investigations 112) - 116).

As long ago as ancient Greek mathematicians such as Pythagoras (circa 580 - 500 BC), and probably longer, people looked at patterns of numbers created by shapes. These, the so-called figurate numbers, are illustrated in the figures below.

119) Show that the numbers formed by the stages in the triangle, the Triangular Numbers, are the same as the Handshake Numbers.

120) Show that the numbers formed by the stages in the square, the Square Numbers, are 1, 4, 9, 16, 25, as we might expect.

When we generated the handshake numbers one way we did this as sums of series. In this series the first difference between consecutive terms is always 1. Namely,

1 = 1 1 + 2 = 3 1 + 2 + 3 = 6 1 + 2 + 3 + 4 = 10

121) Show how the square numbers can be written as sums of series.

123) What are the first differences between the terms that make up the series?

124) You should see a pattern forming. Illustrate this pattern using the Pentagonal Numbers.

125) Repeat 124) to create the Octagonal Numbers.

Perhaps surprisingly, the Natural Numbers 1, 2, 3, 4, 5, can also be formed in this way:

126) Show how the natural numbers can be written as sums of series where the terms are constant with value 1.

127) Using 126) and the geometric patterns investigated above to explain visually why it makes sense to call the natural numbers of the Linear Numbers.

Morgan's Theorem That is perfect for an aside. It seems nicely related to the stuff that is going in here. Why not have it in this section?

5. Concluding Essays

64. Need to have some closing essays. Maybe something about problem solving? It would be nice to have something that forced them to reflect on the readings at the outset. Have them find a quote on problems and/or problem solving.

65. Why were there four problems at the outset?

66. We expect that you used several different strategies to solve the problems 4. If you look back at the strategies that were described 3 there were eight strategies. Is there some value in having so many different strategies? Explain.

6. Further Investigations

The Line Problem can be refined and extended in many ways. Beautiful patterns continue to emerge and the problems range from the level of the Line Problem to areas of open research questions.

F1. Repeat your analysis of the Line Problem by determining how many *unbounded regions* are formed by the lines.
F2. Repeat your analysis of the Line Problem by determining how many *bounded regions* are formed by the lines.
F3. Amoung the bounded regions, different shapes may be formed. Can you determine the types and number of each of these shapes?

Instead of allowing our lines to be placed arbitrarily, we can arrange the lines regularly so they form regular polygons in their center, as shown below.[2]. ???Insert p. 118 figure 11.1 from Pedersen.

F4. Into how many unbounded regions is the plane divided by the lines?
F5. Into how many bounded regions is the plane divided by the lines?

7. Teachers Manual - General

For the Line Problem students who collect data should readily find the following:

$n = L$	P	R
0	0	1
1	0	2
2	1	4
3	3	7
4	6	11
5	10	16

Inductive argument for the points is that each line must intersect each other line in one NEW point. So we have $p_n = p_{n-1} + (n - 1)$.

Inductive argument for the regions is that when a new line is drawn, each time it intersects an old line (or infinity), one region has be dissected into two. It must intersect exactly n-1 lines and infinity once, giving n new regions; i.e. $r_n = r_{n-1} + n$

What should be clear from the algebraic comparison of these recursive relationships or, more likely, from students' observations of the data, is that $r_n = p_{n-1} + 1$ and so again, solving one problem essentially solves the other.

As indicated in the Further Investigations, one can extend these problems in many ways.

One possibility is to consider the different types of regions formed. The number of unbounded regions is exactly 1, 2, 4, 6, 8,..., 2n,... as each pair of lines forms an unbounded region on the "left" and on the "right". While the shapes of the bounded regions can change (with $n = 3$ there is one bounded region, a triangle, and with $n = 4$ there are three bounded regions, 2 triangles and a quadrilateral, however, with $n = 5$ it is possible to have configurations both with and without a pentagon), the numbers of bounded regions is accessible.

It should be clear that patterns pervade this chapter. It is also interesting to note that each of the strategies in 3 is based on a pattern:

[2]Adapted from "Platonic Divisions of Space" by Jean Pedersen in <u>Mathematical Adventures for Students and Amateurs</u>

3.1	$\leftrightarrow$ Numerical Patterns
3.2	$\leftrightarrow$ Matched Pairs Yield Fixed Sums
3.3	$\leftrightarrow$ Risers of Fixed Height; 2 Staircases = 1 Rectangle
3.4	$\leftrightarrow$ Fundamental Counting Principle for Counting
3.5	$\leftrightarrow$ Theorem is a Meta-Pattern
3.6	$\leftrightarrow$ Factors form a Numerical Pattern
3.7	$\leftrightarrow$ All Terms Share Common Factors
3.8	$\leftrightarrow$ Algebraic Description of Pattern

CHAPTER 7

Pascal's Triangle

When I consider the small span of my life absorbed in the eternity of all time, or the small part of space which I can touch or see engulfed by the infinite immensity of spaces that I know not and that know me not, I am frightened and astonished to see myself here instead of there... now instead of then.

Blaise Pascal (French Mathematician, Physicist, and Philsopher; 1623 - 1662)

Nature is an infinite sphere, whose center is everywhere and whose circumference is nowhere.

Blaise Pascal (; -)

When we cite authors we cite their demonstrations, not their names.

Blaise Pascal (; -)

If God does not exist, one will lose nothing by believing in him, while if he does exist, one will lose everything by not believing.

[This line of reasoning has become known as ***Pascal's wager***]

Blaise Pascal (; -)

Certainly need a better title than that.

Good to have lots of stuff about Pascal's remarkable diversity. Stuff about his theoretical computer. Etc. He was a Rennaisance Man - one that in some sense our return to the Arts is trying to encourage.

1. Investigations

1.1. The Circle Problem. We begin with a problem that looks similar to those in the previous chapter:

The Circle Problems On a circle n points are drawn. Lines are drawn which connect each of the points to all of the other points on the circle.

Circle Problem 1: How many lines must be drawn to connect all of the points?

Circle Problem 2: Into how many regions is the circle decomposed by the lines?

67. Solve Circle Problem 1.

68. Collect some data for Circle Problem 2 into a table. Do you see a pattern? If so, make a conjecture which suggests a solution to Circle Problem 2.

69. Use your conjecture from 68 or estimate how many regions you think there will be when there are $n = 6$ points.

70. Carefully draw a circle and locate $n = 6$ points symmetrically around the boundary. Carefully draw lines and determine how many regions will be formed. How does your answer compare to your prediction from 69?

71. Carefully draw a circle and locate $n = 6$ points around the boundary so that when lines are drawn between the points no more than two lines intersect at a single point. How many regions are formed? How does your answer compare to your prediction from 69?

72. How do your answers to 70 and 71 compare? What does this tell you about inductive reasoning?

73. More generally, what do your answers to 70 and 71 tell you about Circle Problem 2? What do they say about the (mis)perception that in mathematics there is always a "correct" answer?

If we are to continue to consider Circle Problem 2 we will need to determine which "pattern" we will focus on. Following the example of the Line Problem in the previous chapter, we will consider the case where no more than two lines intersect at a single point. When lines (or planes or other higher dimensional *linear subspaces*) are arranged in this way they are said to be in ***general position***.

From now on, we will consider only the general position case and we will call this problem **Circle Problem 2(GP)**.

74. Determine how many regions there are for Circle Problem 2(GP) when there are $n = 7$ points. For $n = 8$ points there are $R = 99$ regions. How many regions are there for $n = 9$ points? (Hint: Neglecting the circle itself, the lines can be arranged to form the *complete graph* on 7 and 9 vertices which are easily found online. Question: Why do you think we didn't ask you to do this with $n = 8$?)

75. Can you find a way to describe the "pattern" that solves Circle Problem 2(GP)? Either i) describe your solution, or, ii) articulate the ideas you have tried as well as the difficulties you have encountered.

We will return to this problem/pattern later and see some beautiful ways to describe it.

1.2. Binomial Coefficient Calesthenics.

1.3. Patterns in Pascal's Triangle.

1.4. Polya's Block Walking. Include several of the summation formulas here for them to discover.

Some ideas for the introduction includes:

Polya's blockwalking idea - I remember doing this in a class one time. It was not that successful. It is hard to illustrate how to do this. Maybe do it with Ls and Rs for lefts and rights so the students can keep track of it? Maybe this is one of those things that it is better to let the students do the problem solving on their own. Don't tell them too much how to do it.

Pizza problems - I have always thought that this is nice. Might be good to have them do this for contrast.

Binomial coefficients - Maybe do this as a way to say, "'OMG, I wish I had know how to do this so long ago when I was in an algebra class!!"

In her Mathematics and Music book Chrissi does this counting the number of different rhythms of given count length with a given number of beats!!! Make a note of this in this book.

Also make a note of the dance connection with the number of different ???? of a given number of steps.

The **binomial coefficients** are defined by

$$\begin{matrix} n \\ k \end{matrix} = \frac{n!}{(n-k)!k!}.$$

1.5. Return to the Circle Problem. We've been using binomial coefficients to describe patterns having to do with combinatorics. Circle Problem 2(GP) is about counting regions, and it turns out, with some ingenuity, we can use them to help solve this problem as well. You'll investigate two different ways to do this below.

We'll see later (resp. we saw earlier) that Euler's formula is a beautiful pattern which simply describes the relationship between vertices, edges, and faces of certain polyhedra. While this was a result about solids, it has a natural analogue for planar regions:

Euler's Theorem for Networks For any *simply connected network* the number of vertices (v), edges (e), and faces (f) are related by [v-e+f=1.]

The circles and lines in Circle Problem 2(GP) form just these types of networks and the regions simply correspond to faces. So instead of counting regions, if we can count vertices and edges we can solve the problem.

76. Begin by collecting some data. Namely, use your figures from ??? to complete the table below. Notice that what we have called lines now consist of several edges and we must include the curves between boundary points as edges as well.

n	L	R	v	e
1	0	1	1	1
2	1	2	2	3
3	3	4	3	6
4				
5				
6				

77. The new data is the vertex and edge data. Are there any immediate patterns that you can decipher in this data? Explain.

Let's start with the vertices. There are n around the boundary of the circle. How many others?

78. How are the vertices which are not on the boundary formed?

79. In the Figure below two interior vertices have been named as indicated. 1425 is so named because it is at the intersection of the lines $\overrightarrow{14}$ between points 1 and 4 and $\overrightarrow{25}$ between points 2 and 5. Name all of the other interior points in this manner.

Sometimes one can solve a problem by translating it into a slightly different problem that can more easily be approached. Maybe we can use this naming scheme to find out how many interior vertices are created.

80. The point 1425 can also be named 2541. It has several other names, give as many of these as you can.

81. Describe what all of the names for the given point in 80 have in common.

82. Is the name 1245 an appropriate name for a vertex? If not, why not? If so, which vertex?

83. Given four numbers of vertex labels, in no particular order, how many different verticies can these labels name?

84. Use what you have learned about binomial coefficients and choosing to determine an expression for the number of interior vertices.

85. Return to 74 and check to see whether your expression gives the correct number of vertices when $n = 7$ and $n = 9$.

Now what about the number of edges?

86. Copy Figure ??? above. At each vertex, write the number of edges that is incident on this vertex. The number is called the ***degree*** of the vertex. What do you notice?

87. How does your answer to 86 change if the number of boundary points, n, is changed? Describe this change in detail.

Let's begin counting...

88. Sum the degrees of the interior vertices. Explain why this sum represents the number of terminal points of edges that the interior vertices can accomodate.

89. Sum the degrees of the boundary vertices. Explain why this sum represents the number of terminal points of edges that the boundary vertices can accomodate.

90. What is the total number of terminal points that the vertices can accomodate?

91. Since each edge has a beginning and ending point, you should now be able to determine an expression for the number of edges. Describe and explain your result.

92. Return to 74 and check to see whether your expression gives the correct number of edges when $n = 7$ and $n = 9$.

93. Use Euler's formula and your recent results to solve Circle Problem 2(GP). Leave your expression for the number of regions in terms of binomial coefficients.

94. This is harder and a bit more involved. But for those that would like to prove that the number of regions is as shown in **??**, there is an ingenious scheme which develops unique names for each the regions developed by John H. Conway and Richard K. Guy. The structure of the names allows one to count the number of regions by their names in a straightforward combinatorial way.

Read the outline on pp. 76-79 of The Book of Numbers. Describe this naming scheme in your own words and then provide examples. Explain why each region has a unique name using between zero and four of the numbers $\{0, 1, 2, ..., n-1\}$. Use basic combinatorial arguments to then complete the proof.

$$\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}$$

and show that these answers agree with the collected data. Does the form of this equation suggest why the pattern one would inductive expect does not hold starting at stage $n = 6$?

95. This must include a discussion of why the 5-term expression shows exactly why 2^n fails.

96. This ends with a comparison of the two expressions.

97. More regions stuff goes here.

2. Conclusion

> Linear programming can be viewed as part of a great revolutionary development which has given mankind the ability to state general goals and to lay out a path of detailed decisions to take in order to "best" achieve its goals when faced with practical situations of great complexity.
>
> **George Bernard Dantzig** (American Mathematician; 1914 - 2005)

> This easy-to-state example [how to best assign 70 men to 70 different jobs] illustrates why up to 1947, and for the most part even to this day, a great gulf exists between man's aspirations and his actions. Man may wish to state his wants in complex situations *in terms of a general objective to be optimized but there are so many different ways to go about it, each with its advantages and disadvantages, that it would be impossible to compare all the cases and choose which among them would be the best. Invariably, man in the past has turned to a leader whose 'experiences' and 'mature judgment' would guide the way.*
>
> **George Bernard Dantzig** *(; -)*

Undertanding the way the lines in Fig. ??? intersect the *feasible region* labelled R is the key to finding an optimal solution to a *two-variable linear optimization program.* This program may be the levels of production of two different items competing for scarce resources or the assignment of two jobs to two employees. Indeed, the geometry inherent in this figure is an archetype for linear programming, an contemporary area of mathematics which distinguished science writer and National Public Radio "Math Guy" describes as:

> Perhaps the single most important real-life problem.
>
> **Keith Devlin** (English Mathematician; 1947 -)

.

Suppose we asked you to extend your investigations of the Line Problems and the Circle Problems to Plane Problems and Sphere Problems? Could we encourage you to think about analogous problems in four-dimensional space or even five-dimensional space? Although we can imagine such spaces mathematically and learn a great deal about them (see e.g Discovering the Art of Geometry in this series) this may seem like an dry, academic pursuit.

But in fact, understanding the interaction between *hyperplanes* and *polytopes* in higher dimensions is the fundamental geometry that underlies linear programming. The polytopes may have hundreds of thousands of sides and there may be tens of thousands of *decision variables* giving rise to a problem in tens of thousands of dimensions.

The end result of linear programming? Job statistics. Obaminoes. Airline scheduling. Waste. Give references. (Devlin? H and L).

In addition to the mathematical connections, the confluence of needs, ideas, tools, and personalities that gave rise to linear programming is worthy of attention.

Often called the "Father of Linear Programming," George Bernard Dantzig

Also, this stuff become important because of the availability of computers. So it will be nice to have this in the chapter about Pascal!!!!!!!!!!!!!!!!!!!!! Volker had an interesting idea about how this illustrates how many different things need to come together to solve a problem, make a discovery, change a paradigm. For LP to happen we needed people thinking deeply about geometry, we needed the computers, and we needed a context set by a real problem that was begging for a solution. "Confluence" of ideas, opportunities, needs, and developments. Note about his name - George Bernard for Shaw; parents wanted him to be a writer. Brother named Henri Poincare for a mathematician. Mention the homework problem thing. This is the set-up for Good Will Hunting. See Snopes.com.

Indeed, in 1975 the Nobel Prize in Economic Sciences was awarded to Professor Leonid Kantorovich and Tjalling Koopmans who:

> largely independent of one another, have renewed, generalized, and developed methods for the analysis of the classical problem of economics as regards the optimum allocation of scarce resources.
>
> (; -)

Koopmans was quite distressed that Danzig did not share in the prize. His "great intellectual honesty" in trying to make this right is described by Michel L. Balinski in "Mathematical programming: Journal, society, recollections" from History of Mathematical Programming: A Collection of Personal Reminiscences by J.K. Lenstra, A.H.G. Rinnooy Kan and A. Schrijver, Elsevier, 1991.

3. Further Investigations

F6. One can use the methods of **??** to determine the *degree* of the polynomial that solves Circle Problem 2GP and, with quite a bit of somewhat intense algebra, determine an expression for this polynomial. Using your data from 74 to make a table of 1st, 2nd, 3rd, 4th, and 5th differences. What does this table tell you?

F7. Use algebra to show directly that the two expressions for the number of regions in 96 are indeed equal. Compare the relative merits of your approach here to that in 96.

F8. Now that you have had some experience with some fairly sophisticated counting you might be able to count the number of regions directly. Namely, show that each line after the first creates one new region at the outset and then as many more regions as intersections with previously drawn regions. Then use your knowledge of the number of lines and the number of intersections to count the total number of regions. [1]

4. Teacher's Manual

A very interesting alternative approach to Pascal's triangle is through probability. This approach, in a guided discovery approach similar to ours, is used quite effectively by Harold R. Jacobs in Chapter 8 of Mathematics: A Human Endeavor.

It should be noted that this approach is not only pedagogically interesting, but has a basis in applications. The Hardy-Weinberg Equilibrium Principle is a fundamental principle of population genetics.

[1] This approach, due to Marc Noy, is sufficiently original and noteworthy that it was published in the Mathematical Association of America journal *Mathematics Magazine* in 1996.

It relies on binomial coefficients and the interpretation of these coefficients in *generating functions*. Weingerg is **Wilhelm Weinberg German physician** (???; ??? - w)ho was credited with the simultaneous discovery only many years later in 1943. Hardy is the famous mathematician **G??? H. Hardy ???** (???; w - h)o thought it somewhat absurd that this principle had not been realized prior to his description of it in 1908:

> I am reluctant to intrude in a discussion concerning matters of which I have no expert knowledge, and I should have expected the very simple point which I wish to make to have been familiar to biologists. However, some remarks of Mr. Udny Yule, to which Mr. R. C. Punnett has called my attention, suggest that it may still be worth making...[2]

Punnett is **R.C. Punnett ???** (???; ??? - w)ho invented *Punnett squares* that we all study in high school biology. Punnett and Hardy played cricket together and it is said that Hardy's discovery was made in this setting.

It is instructive to consider the forms of the answers to Circle Problem 2(GP). A quick analysis of the binomial coefficients show the number of regions will be degree 4 in n. Indeed, for those students who have considered 3.5 will be interested to consider the sequence of difference here:

n	R		1st Δ		2nd Δ		3rd Δ		4th Δ
1	1								
		>	1						
2	2			>	1				
		>	2			>	1		
3	4			>	2			>	1
		>	4			>	2		
4	8			>	4			>	1
		>	8			>	3		
5	16			>	7			>	1
		>	15			>	4		
6	31			>	11			>	1
		>	26			>	5		
7	57			>	16			>	1
		>	42			>	6		
8	99			>	22				
		>	64						
9	163								

Hence, students who investigate using this route will see from the Fundamental Theorem of Discrete Calculus that the number of regions is a quartic equation.

However, determining the coefficients is an mechanical, uninformative process that leads us to the algebraic expression for the number of regions given by:

$$R = \left(\frac{1}{24}\right) n^4 - \left(\frac{1}{4}\right) n^3 + \left(\frac{23}{24}\right) n^2 - \left(\frac{3}{4}\right) n + 1.$$

Certainly the expressions using binomial coefficients are more aesthetically pleasing, easier to work with (96), and far more instructive (95).

The data for 76 is:

[2] *Science*, Vol. XXXVIII, July 10, 1908, pp. 49-50

n	L	R	v	e
1	0	1	1	1
2	1	2	2	3
3	3	4	3	6
4	6	8	5	12
5	10	16	10	25
6	15	31	21	51

4.1. Selected Answers. Answer to 83: Each set of four vertex labels uniquely defines an interior vertex.

Answer to 84: The number of interior vertices for any n is $\binom{n}{4}$.

4.2. Bibliography. The link between the Circle Problem and Linear Programming was suggested parenthetically by David M. Bressoud in http://www.maa.org/columns/launchings/launchings_12_07.html . He attributes much of this activity to George Polya from the video Let us Teach Guessing published by the Mathematical Assocition of America.

4.3. Random Notes. http://mathforum.org/library/drmath/view/51708.html http://en.wikipedia.org/wiki/Divic p. 187 of Maurer and Ralston's Discrete Algorithmic Mathematics. Some related ones in the exercises. Seems that it is Bassarear, Chapter 8 - Basic Concepts of Geometry. Exploration 8.4 for example. Did I put any of this into a file somewhere and that is why I do not have it printed out?

CHAPTER 8

Library of Patterns

Throughout this volume we have emphasized the view that mathematics is the science of patterns. We have also tried to help you see more deeply how patterns play a central role in nature, in the arts, in the sciences, and throughout the human experience.

Before continuing with detailed investigations of other patterns, this is a good juncture to pause and celebrate both the centrality and diversity of mathematical patterns in the world around us. We could do this by including a nice ??? of full-color photos. But this would be expensive. More importantly, it would put you in the role of simply an observer of mathematics while we have tried to have you be center of the action in this journey.

So your job at this point is to create a permanent collection of patterns - a **Library of Patterns** if you will. This library of patterns should be considered a central part of this book. Along with your notebook, it is a way of personalizing your mathematical journey.

There are several different ways we can suggest for you to create a library of patterns. Maybe you have your own ideas. Or maybe your guide will have some ideas or requirements.

1. Image Gallery

Throughout we have talked about these explorations as a journey and you as an explorer. Explorers usually come back with pictures, why not here?

Need to flush this out some more. Maybe talk to a few people about it.

Mention digital cameras, cellphones, etc. Should also be allowed to use images that they find elsewhere. It is pretty hard to take a picture of the solar system with the planets shown moving in elliptical orbits.

Again, can do this physically on hallway walls, in a display case or bulletin board, in the classroom.

As they are likely to be all digital, it is perfect for a wwwebsite.

2. Poster Gallery

At our college one way we create a library of patterns is for students to create a gallery of posters, each poster celebrating a different pattern. The posters are hung in Department of Mathematics hallways and classrooms for all to see - including students and faculty from other classes who are equally excited to see the many connections. Each poster hangs for a week or more until it is replaced by another student's poster. Over the course of the semester each student's poster has an opportunity to enlighten all those who pass by it. A sign-up sheet insures that all students choose different pattern. Over the course of the semester this means a general group of explorers (class) can create a gallery of 30 patterns. Additionally, each poster is accompanied by a one-page handout. Explorers collect each of these handouts and together they form a permanent record of our library of patterns.

Such a poster session can also be done electronically with posters posted online allowing the world access to your Poster Gallery of Patterns.

For those of you who have not ever seen a poster session, they are widely used to publicize, announce, and/or present the results of research investigations. They are used in professional conferences (including virtually all conferences for mathematicians and scientists), college and university courses, and meetings of all kinds. They are useful because many posters can be displayed without the time and space limitations that traditional presentations impose. Additionally, it makes it easier for participants to browse and find research of interest.

For each explorer's pattern it the poster should:

- Describe the pattern in detail, using a variety of different representations as appropriate.
- Describe why this pattern is of interest to you and may be of interest to others.
- Describe the importance of this pattern - why it appears, what is signifies, how it evolves, etc.
- Describe how this pattern can be analyzed, represented, applied, adapted, and/or related to other patterns of interest. You are free to choose topics whose mathematical analysis is much more sophisticated than you and I can currently understand - this is a survey poster. Indeed, I would encourage you to chose a topic vivaciously - the more interesting the better.
- Provide several references where the interested reader can find more information about this pattern. This should include not only print and Internet references, but also interactive online programs, methods and/or tools for constructing this pattern physically, museums where the patterns can be observed, etc.
- Include a partial history of the development and/or genesis of this pattern.

To keep everybody involved, each poster is self-reviewed, peer reviewed, and reviewed by the teacher. They are evaluated in five categories, which your group may decide to adjust however you see fit. Our categories are:

- An interesting, engaging, and/or important choice of pattern which adds vitality and breadth to our Library of Patterns.
- An informative and accessible description of this pattern, discussion where the pattern arises, description of the importance of this pattern, why this pattern is of interest to you and others, the pattern's impact, applications of this pattern, and the genesis of this pattern.
- An accessible survey of the mathematical analysis of this pattern, including as many of it's representations as appropriate. This is the more mathematical portion of your poster.
- An appropriate collection of additional information interested readers can use to pursue the topic in greater depth. These may include: book, journal, audio, video, and other media and multi-media citations; Internet resources; reviews; museum holdings; event dates; etc.
- A physical construction of a high quality poster and handout, including: appropriate design, pleasing visual layout, effectiveness, appropriate mix of media and information, effort, etc.
- A useful, interesting, appropriate handout for our library of patterns.

We have found this to be a wonderful way to celebrate patterns. We invite your group of explorers to try it as well.

CHAPTER 9

Circles, Stars, Gears, and Unity

The eye is the first circle; the horizon which it forms is the second; and throughout nature this primary figure is repeated without end. It is the highest emblem in the cipher of the world.

Ralph Waldo Emerson (American Philosopher and Essayist; 1803 - 1882)

The universe stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.

Galileo Galilei (Italian Astronomer, Physicist, and Philosopher; 1564 - 1642)

Symmetry, as wide or as narrow as you define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.

Hermann Weyl (German Mathematician; 1885 - 1955)

FIGURE 1. 19th century Tibetan mandala of the Naropa tradition; Rubin Museum of Art

1. Introduction

One title for this chapter could have been "Man and His Symbols" after the last full-length work [?] of the preiminent psychologist **Carl Gustav Jung** (Swiss psychologist; 1875 - 1961). Written to introduce general readers to his theories, mandalas played a central role.

> I had to abandon the idea of the superordinate position of the ego. ... I saw that everything, all paths I had been following, all steps I had taken, were leading back to

> a single point – namely, to the mid-point. It became increasingly plain to me that the mandala is the centre. It is the exponent of all paths. It is the path to the centre, to individuation. . I knew that in finding the mandala as an expression of the self I had attained what was for me the ultimate.[1]
>
> (; - I)

n addition to the gender issue that we would have to correct in such a title, it leaves out the fact that the aesthetic and mathematical aspects that underly manalas are a phenomena found throughout nature as well.

Figure 2. Snowflake photographs: one of the earliest recorded (circa 1900 by **Wilson A. Bentley American Farmer and Photographer** (1865; 1933 -)) and a modern image (by **Kenneth Libbrecht** (; -))

Throughout this chapter are scattered images that illustrate the diversity of mandalas and mandala-like figures in our world.

Figure 3. Google Earth image of Palmanova, Italy

Such images play prominent cultural, artistic, psychological, and symbolic roles. We invite you to pursue some of these roles, here we will consider their mathematical connections.

[1]From Memories, Dreams, Reflections.

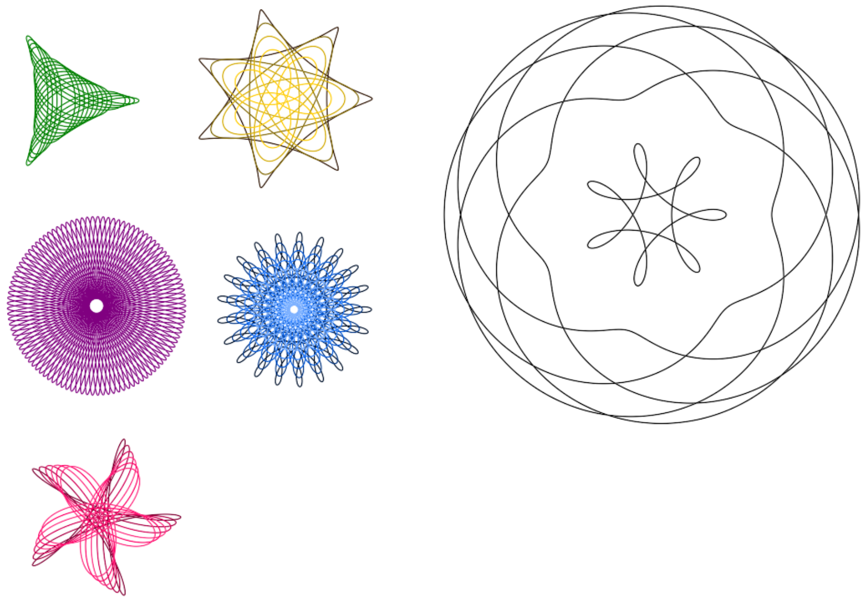


FIGURE 4. Spirograph images

What connects these objects mathematically is the type of symmetry that they exhibit. Each of the images included in this chapter exhibit what is known as *point symmetry*. Objects that exhibit point symmetry are generally called ***rosettes***. The study of these symmetries - and their higher dimensional analogues, *Line Symmetry* whose objects are known as *Friezes* and *Plane Symmetries* whose objects are known as *Tessellations* - are of profound importance. The study of these symmetries is wonderful and we encourage you to consider investigating sometime, perhaps with the help of the fine books [**?**], [**?**], [**?**], and/or [**?**]. Here we will briefly consider point symmetries before focussing on other patterns in specific rosettes.

2. Point Symmetries

Mathematicians are quite particular about their definitions. When one is going to prove things, this is imperative. For this chapter it will be sufficient for you to have an operational understanding of point symmetries. For formal defintions, please refer to the works cited above.

Consider the figure in 5 as a real pinwheel with a fixed center about which the arms may freely rotate. As indicated in the figure, it we rotate the pinwheel by 45 degrees, 90 degrees, 135 degrees,. . . ,360 degrees then the figure will look exactly the same as it did originally. We call this ***rotational symmetry*** and say that the figure has 8-fold rotational symmetry because there are eight different ways it can be rotated and still look the same as when it started.

Consider the figure in 7 with mirrors placed on the dotted lines. If we look at any one of these mirrors the image *reflected* in the mirror looks identical to the part of the figure that is actually obscured by the mirror. We call this ***reflection symmetry*** and say that the figure has 6-fold reflection symmetry because there are six different locations where mirrors can be placed while providing symmetric images.

An object has **point symmetry** if it has either rotational symmetry and/or reflection symmetry.

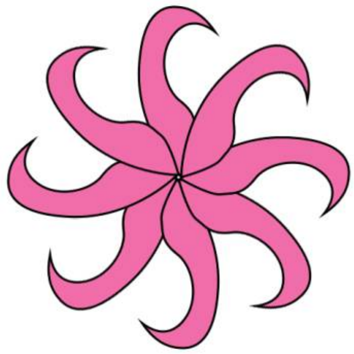


FIGURE 5. A figure with 8-fold rotational symmetry

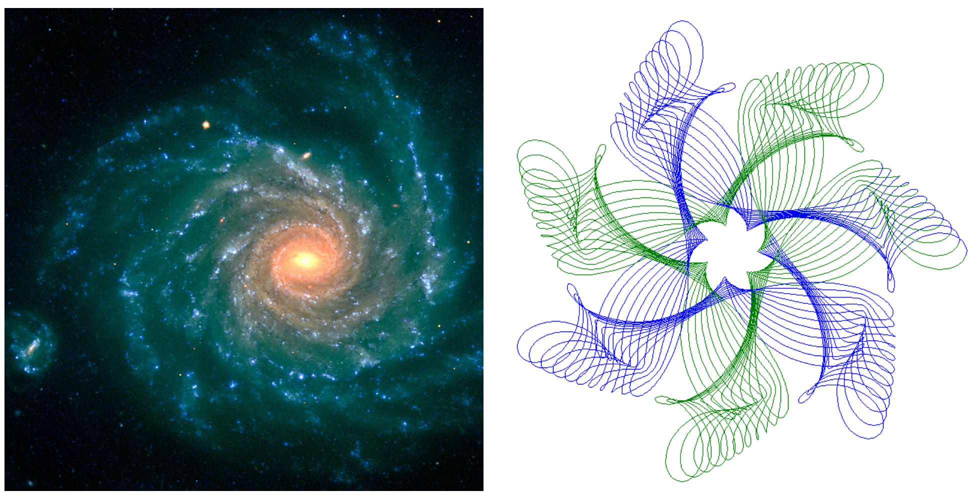


FIGURE 6. Spiral Galaxy NGC 1232 and a Spirograph image with seven-fold rotational symmetry

We are going to focus our attention on particular rosettes - ones which arise through toys.

3. Spirograph and TrippingFest

The rosettes pictured in 4 and on the right of 6 are ***Spirograph drawings*** made from the Spirograph toy. Pictured below, this toy is used by placing a pen in a hole in a geared, circular wheel and then moving the wheel with the pen so it travels around the edge of a geared circular ring or straight bar. Invented in 1965 by **Denys Fisher** (English Engineer; 1918 - 2002) Spirograph was a worldwide phenomena. While low quality variants of Spirograph are still available, unfortunately the original is not available. Used sets can be found on Ebay for several hundreds of dollars.

Spirograph brought great joy to generations of users, allowing them to make wonderful images. It is also a wonderful tool for exploring mathematics and has important historical connections. We will explore many of these connections below. Part of our exploration will be carried out using online applets. Before moving on to the Investigations we would like to make note of a more modern rosette generating tool. . .

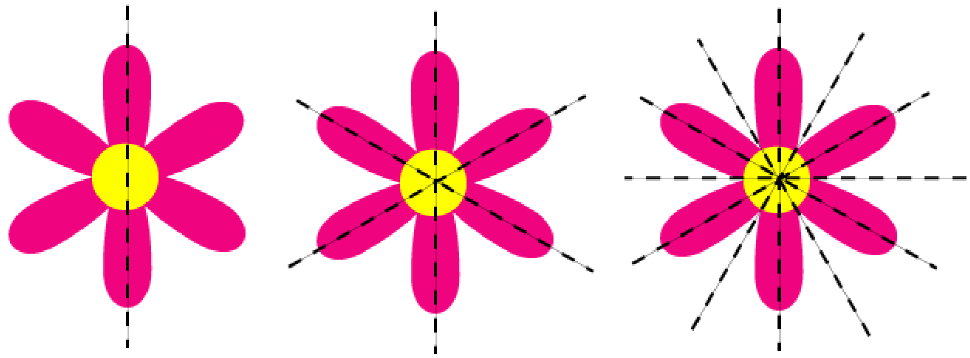


FIGURE 7. A figure with 6-fold reflection symmetry

FIGURE 8. The original Spirograph set

As we write this, **TrippingFest** is one of the top 100 iPod apps. Through a simple interface users can make wonderful art of all sorts, as shown in 9.

If you have access to this app, or have a friend who does, try it out. You can use *Swirl* to make swirls - the archetypes for rosettes whose point symmetry groups consist only of rotations, and are said to have *cyclic symmetry groups*. If we use *Centered Polar* we have the archetypes for rosettes whose

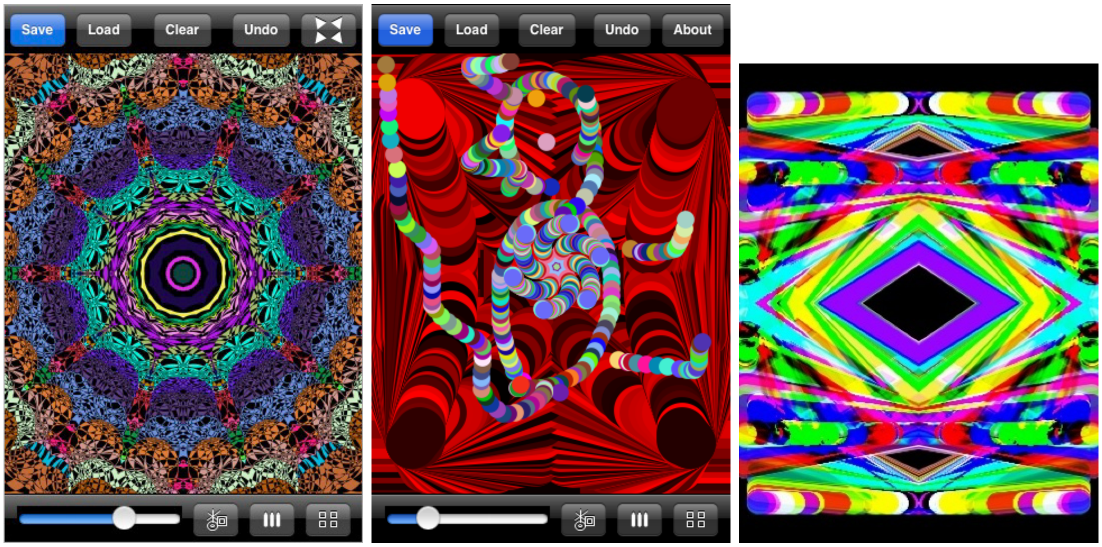


FIGURE 9. A TrippingFest images

point symmetry groups consist of both rotations and reflections, and are said to have *dihedral symmetry groups*. If you do not have access you can see TrippingFest in action on YouTube.[2]

The fact that this app is so popular gives added evidence of our obsession with both symmetry and art.

The genesis of this app also is a wonderful story:

3.0.1. *The Development of TrippingFest.* TrippingFest was developed by **Forrest Heller** (American College Student; -), who describes its development as follows:

"Around 2003, I wrote a drawing program with Visual Basic 5 (not 6). You could draw lots of different patterns with random colors. It had automated random drawing and around 20 different patterns. I was *in middle school at the time*, so it also made fart noises."

"After waiting for someone to make something superior and more complete (and release it for free), I could find no programs that captured the essence of TrippingFest. In early 2009, I embarked on efforts to make a multi-platform drawing platform based on the Visual Basic program with intent to release. I chose Java and before long I had a nice proof-of-concept. When I showed most people the demo they found it amazing. However, I swamped with irrelevant college coursework and I had to discontinue serious development. After school ended, I realized I would never finish the desktop version: I would never stop adding features."

"I then realized that people who have iPhones would probably enjoy the app. I personally don't own an iPhone. I contacted some people over the Internet and found two people willing to loan me equipment for developing an iPhone application. Alex Alba lent me his iPhone for a week (In exchange, he got to use my really old Nokia brick phone). Kyle Coe/Sue Coe lent me their MacBook Intel with Leopard."

"*In one week*, between work and school, I learned Objective-C, learned how to use the iPhone SDK, and finished the iPhone version of TrippingFest. Ironically, I still don't and probably never will own an

[2] http://www.youtube.com/watch?v=-Yp3m2hPvjk&feature=related

iPhone."[3]

4. Investigations

1. Suppose an object has rotational symmetry. When the object is rotated via one of these symmetries do some points in the object remain fixed during the rotations? If so, which ones?
2. Suppose an object has reflective symmetry. When the object is reflected via one of these symmetries do some points in the object remain fixed during the reflection? If so, which ones?
3. Do you think point symmetries is an appropriate name? Explain.
4. Find a way to classify the figures in 2, 3, 6, 4, 9, 11, 12, 18, and 10 by separating them into groups that you think help organize the different types of point symmetries that they illustrate. (Please ignore minor variations and background clutter, concentrating on the larger scale makeup of the images.) Describe your classification scheme.
5. Find a dozen objects that are of interest to you and who exhibit point symmetry . Describe them and then draw them - at least schematically.
6. Classify the objects in 5 as you did in 4.

4.1. Star Polygons.

Unc, why do they call them stars when they are really round?

K.C. Fisher (American child at age 2; 1983 -)

For these investigations you will have to enlist the help of peers and will need quite a bit of string, rope, or ribbon.

In the figure ??? 5 people have formed a circle. Each person is connected to the person next to them with a length of rope. Notice that the rope forms the shape of a *regular pentagon*.

7. Have five people form a circle. Connect every *other* person together with rope. (I.e. the first to the third to the fifth to the second, etc.) Describe the resulting shape formed by the rope and draw this shape.

The figure you created in Investigation 7 is called a *star polygon* - it is a star and, like a polygon, it is created by line segments that have been connected end to end. We will use the term ***star polygon*** to refer to all objects formed in this way, including the pentagon.

We would like to give the different star polygons names to identify specific star polygons. You should see that there are two key features in constructing them, the number of people and the count between people holding a segment of rope. Our notation to identify star polygons will be

$$\binom{\text{Number of People}}{\text{Count between people}}$$

E.g. the pentagon is $\binom{5}{1}$ and the figure in Investigation **7** is $\binom{5}{2}$. If you had 7 people and every 4^{th} was connected by rope the star polygon would be named $\binom{7}{4}$.

So there is consistency, there are a few other terms we would like to define so they can be used in a standard way. We will call star polygons like $\binom{5}{2}$ =7***regular*** because they can be made from a single length of string. Star polygons like $\binom{6}{4}$ = 0will be called ***compound*** because they require more than than a single length of string to form. We will say 0has two ***components*** which are the two $\binom{3}{1}$ star polygons $\triangle$ and $\triangledown$. Star polygons like $\binom{6}{3}$ = Cwill be called ***degenerate*** as they are not polygons.

The structure of much of this book is guided discovery. Sometimes you will need more guidance than others. Here you should begin to see patterns very quickly on your own. So we are going to leave you to pursue this exploration on your own quite a bit.

[3]From http://www.forrestheller.com/drawing/ Emphasis added.

At the end of the chapter are template which show the *roots of unit* that may be useful in drawing your stars.

INDEPENDENT INVESTIGATION 1 - A GALAXY OF STARS Create 30 different star polygons. Name each using the naming convention above and record an image of the object. For each, determine whether they are regular, how many components they have, whether they are degenerate, and any other interesting properties they have.

Now that you have a large collection of star polygons, you should look for patterns. As you find these patterns you may have to create other examples to test your patterns and conjectures. This search might take you as long as 15 guided Investigations usually do, maybe even a whole class period or more.

INDEPENDENT INVESTIGATION 2 - FINDING STAR POLYGON CONSTELLATIONS Using the star polygons you have just created, find 5 different patterns that help organize the star polygons into "constellations" of similar objects. Please be precise in the description of your patterns, using appopriate notation and providing examples as illustrations as apppropriate.

You are working like a mathematician now, discovering all sorts of interesting patterns, making observations, and discovering hidden connections; this is good. It is what mathematicians do.

But there is something you need to know about mathematicians - once they begin an investigation like this they are not happy until they have a complete characterization. To have a ***complete characterization*** a mathematician must be able to describe the ojects in their entirety, organizing the entire system, capturing all of the possible mathematical features, and being able to determine exactly what the outcome will be.

This is what you should now turn to. It will take quite a while to complete this.

INDEPENDENT INVESTIGATION 3 - A CALCULUS OF STAR POLYGONS Completely characterize the family of star polygons $\binom{n}{d}$.

You're back. Most of our students make quite good progress for a while, but because this is a fairly big task they often need some help. In particular, they often ask, "How do know when I am done and things are completely characterized?" Below are a few questions. If you can answer them using your characterization, you are probably done:

QUESTION 1. Describe all of the star polygons $\binom{13}{d}$. How many are there? How many are regular? Degenerate?

QUESTION 2. How many distinct regular star polygons $\binom{50}{d}$ are there? For those star polygons $\binom{50}{d}$ that are compound, how many components can they have? For each different number of components, how many distinct star polygons are?

QUESTION 3. Suppose we repeated the questions in 2 but asked you to answer these questions for star polygons $\binom{7917}{d}$. Without doing it, describe what you would need to do to answer these questions. What properties of the number 7917 are important?

Having completed these significant investigation you might wonder about the title of the last one. The term *calculus* is often used to refer to the mathematical fields of *differential calculus* and *integral calculus* that were independently invented by **Isaac Newton English Mathematcian and Physicist**

(1642; 1727 - a)nd **Gottfried Wilhelm Leibniz German Mathematician and Philosopher** (1646; 1716 - a)nd which remain one of the most fundamental descriptive tools used by human beings to describe change over time. When you hear about AP Calculus in high school or Calculus in college, this is what is being referred to. So why here?

The word calculus actually has a broader meaning - "a method of computation or calculation within a symbolic system." This meaning comes from the Latin root *calculare* which means to calculate. One uses the phrase *the* calculus to differentiate the field made up of differential and integral calculus, which is considered in Discovering the Art of Calculus, from other calculi such as *propositional calculus* which is a branch of mathematical logic.

FIGURE 10. Sand Dollar and Sea Urchin Shells

FIGURE 11. Flowers on the island of St. Croix

4.2. Spirograph Investigations.

As the small pebble stirs the peaceful lake;
The centre mov'd, a circle straight succeeds,
Another still, and still another spreads.

Alexander Pope (; -)

Introduced in Section 3, Spirograph is wonderful, simple tool for making art. The purpose of this section is to discover the mathematical properties that are inherent in this tool and its images. It is helpful to have a Spirograph and not just utilize the online applets, so see if you have one at home, can borrow one from a friend, or can share with some of the other explorers in your group.

Figure 12. Mandala tatoo

Begin by playing with a Spirograph or seeing how one works in action if you don't have a physical version. There are a number of videos available on YouTube and other video sharing sites that you can check out.[4] A still picture is below in Figure 13.

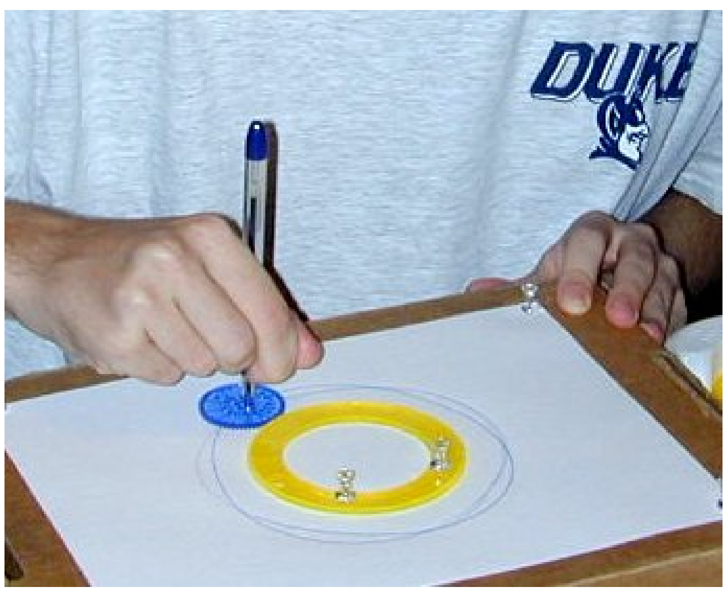


Figure 13. A Spirograph in action

You will also need access to applets that mimic Spirograph. There are hundreds available free online. Find your own or use one the ones below:

[4]An original, 1970's era ad for Spirograph is available at http://www.youtube.com/watch?v=jhW0GE95fMI&feature=rec-LGOUT-exp_fresh+div-1r-3-HM under the name "Spirograph Commercial". Also on YouTube is a commercial for a recent knock-off, ThinkGeek's "Hypotrochoid Art Set" at http://www.youtube.com/watch?v=Q8ZQ2INGvEw .

`http://cgibin.erols.com/ziring/cgi-bin/spiro.pl/spiro.html`
`http://wordsmith.org/~anu/java/spirograph.html`
`http://thinks.com/java/spiro/spiro.htm`

WARNING
The majority of online sites prescribe the size of the wheels and rings by their radii. Following the lead of the original Spirograph, we will refer to wheels and rings by the number of teeth. As you use the applets we urge you to think about the specification of circle sizes in terms of teeth instead of radii so we are using a consistent language. (Question to Ponder: Why is it ok to do this? I.e. why won't this change our observations?)

The **Spirograph figures** we will make here will all be made with *circular wheels* which roll along the inside of *circular rings*. The gears on these wheels and rings are uniformly spaced and will remain in contact at all times so there is no slipping or skipping. The figure will not be considered complete until our pen has returned to its exact starting point. The formal mathematical name of such figures are **hypotrochoids**.

We will follow the original notation for the number of teeth on these pieces: 30 denotes a wheel with 30 teeth. The circular rings have teeth on the inside as well as the outside. We'll only use those on the inside. 105 denotes a ring with 105 teeth on the inside of the ring.

As you work you should clearly label the wheel, ring and hole numbers you use to make figures as they will be critical to the analysis.

8. Choose one ring and one wheel. With your pen in any one of the holes, draw a Spirograph figure. Describe its symmetries.
9. Using the same ring as in 8, draw a Spirograph figure with your pen in a different hole. Describe its symmetries and compare it to your previous figure.
10. Using the same ring as in 8, draw a Spirograph figure with your pen in a different hole. Describe its symmetries and compare it to your previous figures.
11. Based on Investigations 8 - 10, what effect does the choice of hole seem to have on Spirograph figure that is drawn given a specific ring and wheel?
12. If you wanted to draw a Spirograph figure that resembled the star polygons considered above, what would the most appropriate choice of hole be?
13. Draw a Spirograph figure with the ring 96 and wheel 48. What is the result?
14. Now draw a Spirograph figure with the ring 96 and wheel 32. What is the result?
15. Now draw a Spirograph figure with the ring 96 and wheel 24. What is the result?
16. You should see a pattern forming. Using this pattern, what do you think would happen if you used wheel 16? Wheel 12?
17. What other size wheels could be used to illustrate the pattern you described in 16? Explain.
18. Describe the physical characteristics of the Spirograph that explain why the pattern you have described in Investigations 16 and 17 occurs.
19. Notice that the "petals" that make up the Spirograph figures considered in Investigations 13 - 18 are drawn in order, one after another, in a circular order and the figure is complete after one full rotation of petals have been generated. Using the inside of the 105 ring, describe the Spirograph figures whose petals are generated in this same way and identify explicitly the wheels that would be necessary to make each of them.

4.2.1. *Epicycles and Astronomy.* In Discovering the Art of Mathematics - Student Toolbox we described ways that the question "when will I ever use this?" is such an inherently limiting question.

While the caveats we gave there hold just as well for Spirograph figures and star polygons that we have been considering here, we would be remiss if we did not note a direct connection between Spirograph figures and the way in which people understood the universe in which we live for *thousands of years*.

The ***Pythagoreans***, the cult-like followers of **Pythagoras Greek Mathematician and Philosopher** (ca. 570 B.C.; ca. 495 B.C. - ,) believed that the moon, planets, and stars were carried around on a giant crystal sphere. In their motions they emitted differnet tones based on their locations. All told, they created a *music of the spheres*. Such a romantic vision of the heavens was compatible with the central role that both music and number played in their philosophy. This view extended nearly two centuries, all the way through **Johannes Kepler German Astronomer and Mathematician** (1571; 1630 - w)ho used the five *Platonic solids* as carriers of the planets:

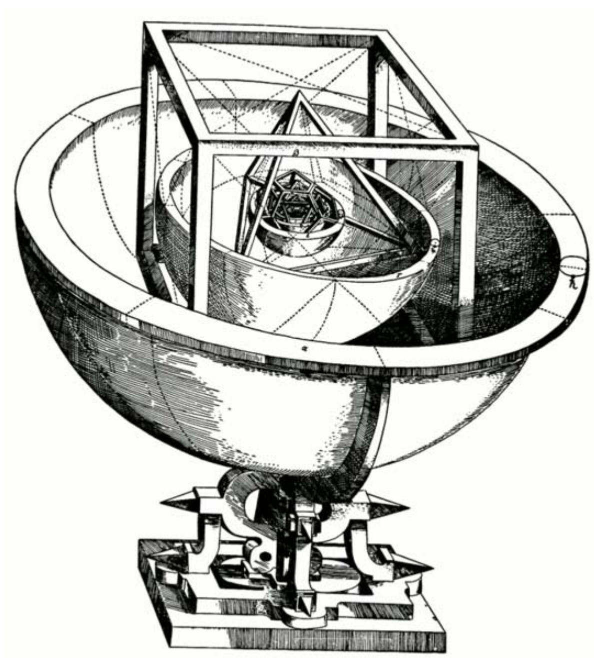


Figure 14. Kepler's model of the solar system using the five Platonic solids

Yet careful observers of the night sky realized that important objects exhibited ***retrograde motion*** where they would occasionally move in the opposite direction in the sky than the normally did. This could not be explained by any fixed sphere model of the heavens nor of routine circular orbits of the heavenly bodies about a central Earth. Trying to keep the Earth at the center, new models were needed.

The Almagest by **Cladius Ptolemy** (Greek Astronomer and Mathematician; 90 - 168) is one of the most important books on astronomy ever written. It laid out a system of *epicycles* which described the retrograde motion that had been observed. These epicycles are formed by one wheel rolling along another with the planet located on the smaller wheel. The paths of the heavenly bodies were *exactly* predicted to be Spirograph figures. Viewed from our perspective from Earth, such epicycles approximated retrograde motion as well as the limited observations allowed. Many interesting applets which illustrate both epicycles and retrogade motion are available online.[5]

[5]E.g. http://jove.geol.niu.edu/faculty/stoddard/JAVA/ptolemy.html and http://csep10.phys.utk.edu/astr161/lect/retrograde/aristotle.html

If you felt that there was something very "orbital" feeling about Spirograph, you are in good company. This was the prevailing model to describe our universe from the time near the birth of Christ until **Nicolas Copernicus Polish Astronomer, Mathematician and Scholar** (1473; 1543 - r)evolutionized our view of the solar system in his publication of <u>De revolutionibus orbium coelestium</u> in 1543.

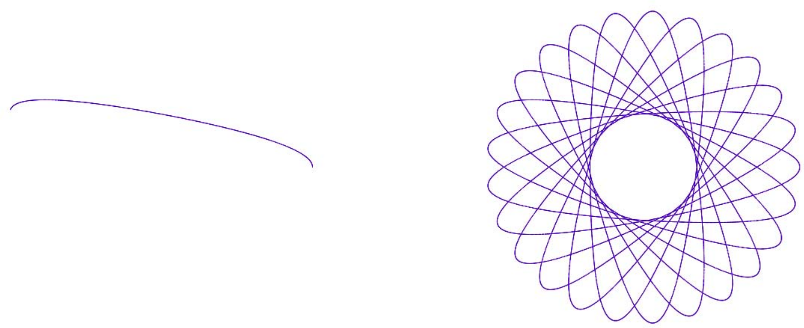


FIGURE 15. An arc of a Spirograph figure from apex to apex, left, and the entire figure, right

4.3. More Spirograph Investigations. By the **apex** of a Spirograph figure we will mean a point where the curve is closest to the ring used to create it. Figure 15 shows an arc of a Spirograph curve from one apex to the next.

20. Create the Spirograph figure generated by wheel 40 using the 105 ring. Describe its symmetries. Label the "top" apex 1 and then continue around clockwise numbering the apexes $2, 3, 4, \ldots$ until you return again to the top.

21. Starting at apex 1, follow the Spirograph curve until you reach the next apex that the curve hits. What number apex is it?

22. Continue along the Spirograph curve to the next apex that the curve hits. What number apex is it?

23. Continue along the Spirograph curve to the next apex that the curve hits. What number apex is it?

24. You should see a pattern in Investigations 20 - 23. Describe it precisely. Does this pattern continue? Explain.

25. In Investigations 21 - 23 you followed along the Spirograph curve from one apex to the next apex that this curve hit. Suppose you replaced each of these Spirograph arcs with a line. What figure would be formed? Do you have a precise name for this figure?

26. Create the Spirograph figure generated by wheel 42 using the 96 ring. Describe its symmetries. Label the "top" apex 1 and then continue around clockwise numbering the apexes $2, 3, 4, \ldots$ until you return again to the top.

27. Starting at apex 1, follow the Spirograph curve until you reach the next apex that the curve hits. What number apex is it?

28. Continue along the Spirograph curve to the next apex that the curve hits. What number apex is it?

29. Continue along the Spirograph curve to the next apex that the curve hits. What number apex is it?

30. You should see a pattern in Investigations 26 - 29. Describe it precisely. Does this pattern continue? Explain.

31. In Investigations 27 - 29 you followed along the Spirograph curve from one apex to the next apex that this curve hit. Suppose you replaced each of these Spirograph arcs with a line. What figure would be formed? Do you have a precise name for this figure?

> Before I was two years old I had developed an intense involvement with automobiles. The names of car parts made up a very substantial portion of my vocabulary... Years later... playing with gears became a favorite pastime I became adept at turning wheels in my head and at making chains of cause and effect... Working with differentials did more for my mathematical development than anything I was taught in elementary school.[6]
>
> **Seymour Papert** (South African Mathematician and Educator; 1928 -)

You should be seeing an important connection between Spirograph figures and star polygons. Use this pattern to:

32. Describe the symmetries and the identity of the star polygon which corresponds to the Spirograph figure generated by the wheel 60 and the ring 105 without actually drawing it first. (Feel free to draw it to check.)

33. Determine the wheel and ring used to create the Spirograph curve in Figure 16.

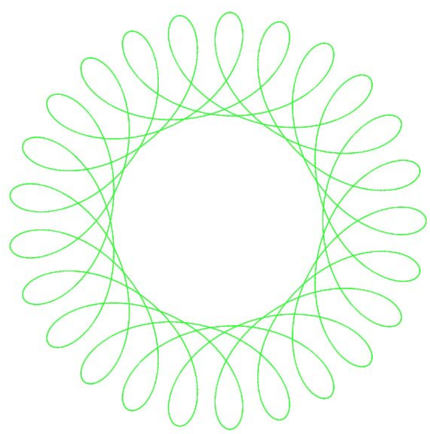


FIGURE 16. Spirograph figure for Investigation 33

(Note: Here one can only determine the relative sizes of the wheel and ring as there are infinitely many different combinations that can make this figure.)

34. Describe the symmetries and the identity of the star polygon which corresponds to the Spirograph figure generated by the wheel 63 and the ring 96 without actually drawing it first. (Feel free to draw it to check.)

35. Determine the wheel and ring used to create the Spirograph curve in Figure 17.

[6]From "The Gears of My Childhood," preface to Mindstorms: Children, Computers and Powerful Ideas.

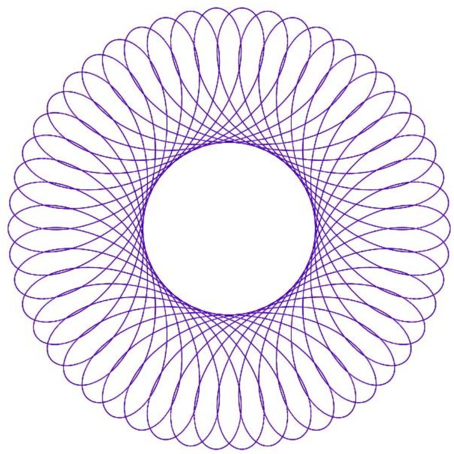


Figure 17. Spirograph figure for Investigation 35

(Note: Here one can only determine the relative sizes of the wheel and ring as there are infinitely many different combinations that can make this figure.)

36. Write a summary which precisely describes in a detailed. quantitative way how one navigates between the world of Spirograph figures and the world of star polygons.

Figure 18. A water drop

4.3.1. *Famous Curves.* Above we saw historically important connections between wheels travelling along wheels, like Spirograph curves, and astronomy. If we expand the type of objects that interact, to not only cirlces, but lines, and squares, we get many other critical links to the history, science, architecture, and art.

During the *Scientific Revolution*, roughly 1540 - 1730, mathematicians routinely challenged other mathematicians and held problem solving competitions. One important challenge is the *brachistochrone problem* which was stated eloquently by **Johann Bernoulli Swiss Mathematician** (1654; 1705 - i)n June, 1696 in the journal *Acta Eruditorum* :

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.
>
> (; -)

The problem is most easily described as a physical experiment. We have two points A and B in space and we wish to build a ramp so we can start a ball at A and have it roll to B.

Brachistichrone Problem: Of all such ramps, what is the profile of the ramp which will enable the ball to reach point B most quickly in a frictionless environment?

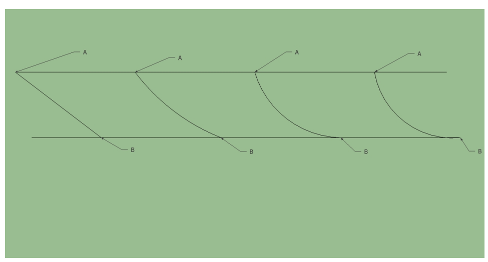


FIGURE 19. Ramps for the Brachistichrone problem

Skiers and skateboarders will have a sense that the latter profiles in Figure 19 will offer dramatically faster routes to the bottom. But what is the optimal profile?

This problem was stated not long after the independent invention of calculus by Newton and Leibniz. This development was a profound change to mathematics and science, playing a central role in the Scientific Revolution. One of calculus' main strengths was its ability to solve maxima and minima problems. Until now such solutions were to find a point in time, a speed, an optimal solution to a single problem with a fixed set-up. The Brachistichrone problem was different, the goal wasn't to find one time, one point, or one speed along a given curve, but to consider the optimal solution among all possible curves that can exist. It solution, and the subsequent blossoming of the methods used to solve similar problems, gave rise to the *Calculus of Variations* which is of fundamental importance in many areas of mathematics, physics, optics and engineering.

Said **Karl Menger** (Austrian Mathematician; 1902 - 1985) of the Calculus of Variations:

> Mathematicians study their problems on account of their intrinsic interest, and develop their theories on account of their beauty. History shows that some of these mathematical theories which were developed without any chance of immediate use later on found very important applications. Certainly this is true in the case of calculus of variations:

> If the cars, the locomotives, the planes, etc., produced to-day are different in form from what they used to be fifteen years ago, then a good deal of this change is due to the calculus of variations.
>
> (; -)

Returning to the problem, the optimal profile is what is now called a ***cycloid***, the curve appearing on the far right in the figure above. But what curve is this? How is it generated? It is the curve traced out by a point on a wheel moving along a straight line. In other words, it is a Spirograph curve made on a straight rod!!

The Brachistichrone problem was solved by Newton, **Jacob Bernoulli** (Swiss Mathematician; 1667 - 1748) (Johann's younger brother), Leibniz, and **Franois Antoine Marquis de L'Hopital** (French Mathematician; 1661 - 1704). Mathematical prowess was strongly valued, for both intellectual and practical reasons. When the solutions were published, Johann prefaced them by noting the solutions showed:

> the three great nations, Germany, England, France, each one of their own to unite with myself in such a beautiful search, all finding the same truth.
>
> (; -)

There is rich history and deep applications related to curves like: the *cardiod*, the *lemniscate*, and the *Witch of Agnesi*. We close with one more, the *catenary*.

A ***catenary*** is the shape taken by a weighted chain or rope under the influence of gravity. Catenaries are fundamental in the architectural design of arches, the St. Louis Gateway Arch, pictured in Figure 20 being a classic example.

FIGURE 20. The St. Louis Gateway Arch

While **Gerson B. Robison** (???; 1909 -) was "picking up my small son's toy blocks, [he] became intriuged with the possibility of finding a cylindrical surface upon which a plank would roll in neutral equilibrium."[7] He discovered that the required curve is a catenary. And if we put a number of catenaries together end to end we get the wonderful square wheeled bicycle pictured in Figure 21 which was invented

[7]From "Rockers and Rollers", *Mathematics Magazine*, vol. 33, no. 3, Jan.-Feb. 1960, p. 139.

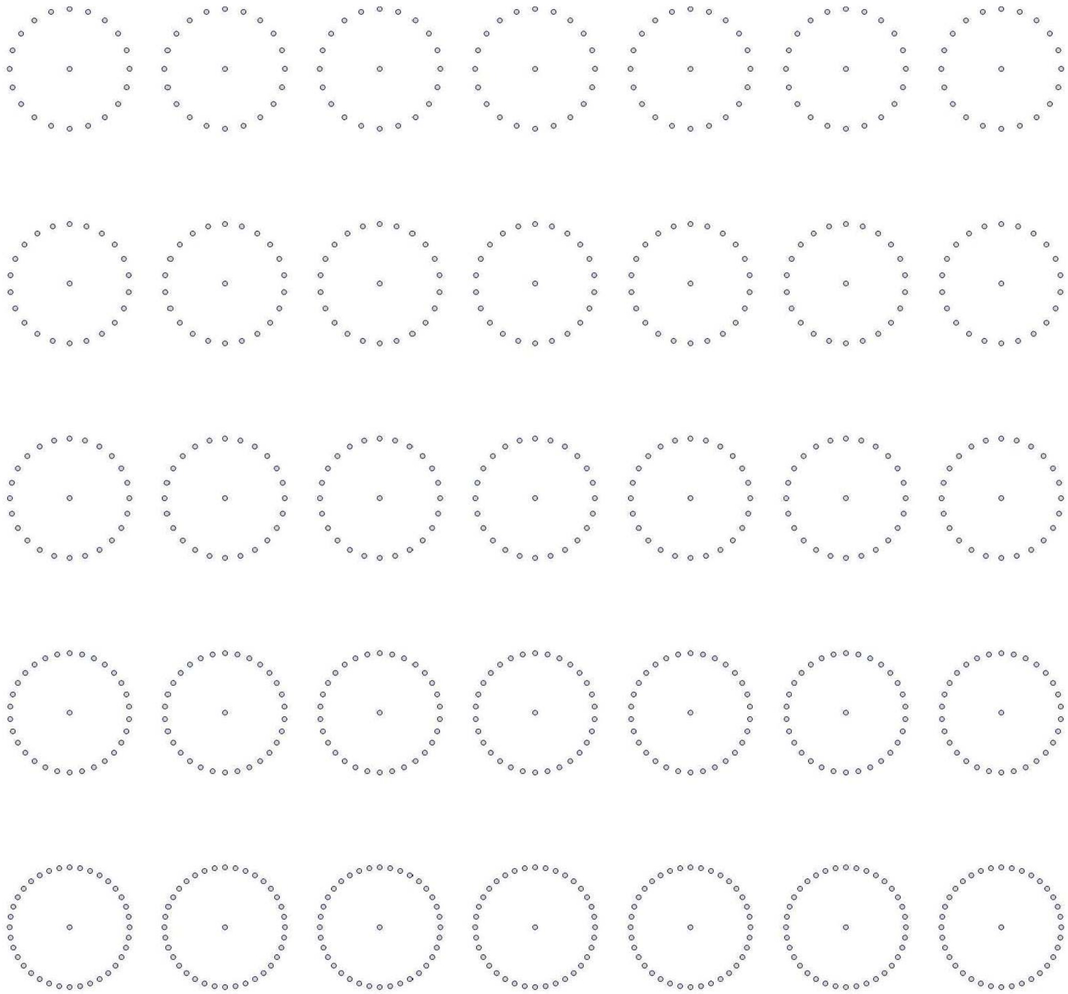
by **Stan Wagon ??? Mathematician** (; - .) The bike rides smoothly, as if it was a normal bicycle as the axles of the square wheel travel in a straight line. With Spirograph figures circles and rings made a famous curve. Here the famous curves coupled with a square bring us back to a line.

FIGURE 21. Stan Wagon on the square wheeled bicycle he invented at Macalester College

We hope you find this interesting. But the principles at work are much more than a curiousity. Many people (e.g. see Building the Great Pyramid in a Year: An Engineer's Report by **Gerard C.A. Fonte** (??? Engineer; -)) have argued that relics found near the pyramid suggest that the huge granite blocks used to construct the pyramids were rolled on catenaries. Fonte, a small man, regularly demonstrates this principle in action by rolling two ton concrete blocks along the catenaries set up in his yard.

5. Roots of Unity Templates

6. Further Investigations



F1. The Kakeya problem is a famous mathematical problem with a surprising solution. Find out about this problem. Then i) describe the problem in your own words, ii) describe the "answer" to this problem in your own words, and iii)

F2. If you were going to investigate the Kakeya problem on your own, are their physical, visual, and/or artistic ways you might do so? Explain.

F3. For those with access to an iPod...] Play around with the iPod app TrippingFest. Can you recognize some other mathematical or artistic issues that are involved other than those we considered involving point symmetry?

se with access to an iPod... Check out Tricky Line in TrippingFest. Is there some connection to the Kakeya problem in some sense? If so, describe it.

se with access to an iPod... How is the Tricky Line app related to the discussion of geometric objects moving around in space above?

7. Teachers Manual

7.1. Star Polygons. Historically star polygons were studied when what we have called point and skip numbers were relatively prime. Unfortunately, this excludes typical important objects like the Star of David which was subsumed under label "star figures." It seems more appropriate to call all of these things star polygons and instead use the adjectives "regular" and "compound" to distinguish the different types. (Following Kinsey and Moore, p. 74.)

The definition in the text using ropes is informal, but can be formalized readily. Namely, to construct $\binom{n}{d}$ you connect every d^{th} point with a line segment. This suggests n pieces of rope, but a few experiments will show that the result is the same as using a length of rope for each orbit.

The star polygons can then be completely characterized based on basic number theoretic properties related to n and d. We describe this characterization briefly here.

For any values of n and d one has:

$$\binom{n}{d} = \binom{n}{d \bmod n}$$

by the very natural of modular arithmetic. Hence, we can assume without loss of generality that $0 \leq d \leq n$.

$d = n$ and $d = 0$ yield nothing.

$d = \frac{n}{2}$ is degenerate, yielding an n pointed star from $\frac{n}{2}$ line segements joined at a single point. While interesting, it is not a polygon.

For any values of n and d one has:

$$\binom{n}{d} = \binom{n}{n-d}.$$

This symmetry and the earlier results insure that to characterize the star polygons we need only consider d with $1 \leq d < \frac{n}{2}$.

One can then characterize the star polygons for a fixed value n as follows:

THEOREM 1. *$\binom{n}{d}$ is a regular star polygon if and only if $\gcd(n,d) = 1$. Moreover, the number of distinct regular star polygons of degree n is $\frac{\phi(n)}{2}$.*

EXAMPLE 1. For $n = 13$ the star polygons in the non-degenerate cases are all regular. The distinct star polygons are $\binom{13}{1}, \binom{13}{2}, \binom{13}{3}, \binom{13}{4}, \binom{13}{5}$, and $\binom{13}{6}$.

THEOREM 2. *$\binom{n}{d}$ is compound star polygon consisting of $\gcd(n,d)$ regular star polygons of the form $\binom{\frac{n}{\gcd(n,d)}}{\frac{d}{\gcd(n,d)}}$ if and only if $\gcd(n,d) \neq 1$. Moreover, the number of distinct compound star polygons of degree n is $\left(\lfloor \frac{n}{2} \rfloor - \frac{\phi(n)}{2}\right)$.*

Here ϕ is the **Euler phi function** from number theory which for each n yields the number of numbers less than n that are relatively prime to n. $\lfloor n \rfloor$ is the **floor function** whose value is the largest integer less than or equal to n for each value of n.

EXAMPLE 2. For $n = 50$ the distinct regular star polygons are:

$$\binom{50}{1}, \binom{50}{3}, \binom{50}{7}, \binom{50}{9}, \binom{50}{11}, \binom{50}{13}, \binom{50}{17}, \binom{50}{19}, \binom{50}{21}, \text{ and} \binom{50}{23}.$$

For the star polygons with multiple components we will write $\binom{50}{10} \cong 10\binom{5}{1}$ to signify that $\binom{50}{10}$ is made up of 10 identical $\binom{5}{1}$ regular star polygons. With this notation the distinct star polygons with two components are:

$$\binom{50}{2} \cong 2\binom{25}{1}, \binom{50}{4} \cong 2\binom{25}{2}, \binom{50}{6} \cong 2\binom{25}{3}, \binom{50}{8} \cong 2\binom{25}{4}, \binom{50}{12} \cong 2\binom{25}{6},$$

$$\binom{50}{14} \cong 2\binom{25}{7}, \binom{50}{16} \cong 2\binom{25}{8}, \binom{50}{18} \cong 2\binom{25}{9}, \binom{50}{22} \cong 2\binom{25}{11} \text{ and} \binom{50}{24} \cong 2\binom{25}{12}.$$

The distinct star polygons with five components are:

$$\binom{50}{5} \cong 5\binom{10}{1} \text{ and} \binom{50}{15} \cong 5\binom{10}{3}.$$

The distinct star polygons with ten components are:

$$\binom{50}{10} \cong 10\binom{5}{1} \text{ and} \binom{50}{20} \cong 10\binom{5}{2}.$$

Notice the wonderful wealth of patterns!

Notice also that what we are doing here is very much like what we would do in reducing fractions to lowest terms.

The topic of symmetries, including the study of Point, Line, and Plane Symmetry (i.e. Rosettes, Frieze Patterns, and Tessellations) as aluded to in the text is a staple of many Mathematics for Liberal Arts texts and does not need to be revisited here. There are several good resources which have a significant inquiry flavor. In order of decreasing depth we recommend the following:

Groups and Symmetry: A Guide to Discovering Mathematics, Mathematical World, Vol. 5, by David W. Farmer, American Mathematical Society, 1995.

Symmetry, Shape and Space: an Introduction to Mathematics Through Geometry by L. Christine Kinsey and Teresa E. Moore, Key College Publishing, 2002.

Geometry: An Investigative Approach by Michael Serra, Key Curriculum Press, 2003.

Have to include stuff about the angles and doing this in Logo. Maybe these things should be in the Further Investigations section. That way students can see them and will have them if the teacher wants to use them. The solutions can be in the Teachers Manual.

7.2. Selected Solutions. In 6 lines are now moving around freely in space as opposed to points on circles moving along fixed geometric objects.

7.3. JF Notes. Links to Kumihamo!!!!!!! This is what links spirograph and star polygons to the Chinese Remainder Theorem via a real-world lift!!

The Islamic artist was not only versed in mathematics in the geometrical sense, but mathematics was integral to his art as it was a 'universal' structure supporting the intuitive insights that characterize all true art. Keith Critchlow, from the introduction to Islamic Patterns: An Analytical and Cosmological Approach.

There must be some definite cause why, whenever snow begins to fall, its initial formations invariably display the shape of a six-cornered starlet. For if it happens by chance, why do they not fall just as well with five corners or with seven? Why always six? Johannes Kepler, from On the Six-Cornered Snowflake

http://vodpod.com/watch/986833-dekepod-episode-005-spirographs-on-steroids Spirograph-like images in Illustrator.

http://thinks.com/java/spiro/spiro.htm

All done in terms of radii. No teeth.

Same one, but with all of the documentation and downloads so you can put it on your site at: http://wordsmith.org/ anu/java/spirograph.html http://www.math.psu.edu/dlittle/java/parametricequations/spirograph/SpiroGraph1.0/index.html This one shows the circles moving!!!!!!!!!!!!!!!!! Also only has radii. And only does external rings.

This is a very cool one that shows the circles moving too!! http://perl.guru.org/lynn/apps/index.html

Wow!! This makes amazing pictures: http://michelle.esfm.ipn.mx/ mrspock/spiro2/ But it is hard to know what it is doing exactly.

From Voltaire's Riddle by Andrew Simoson, MAA, 2010:

(p. 163): Drer's Hypocycloid

Albrecht Drer, the great Renaissance German artist, is credited with being the first to introduce the hypocycloid curve along with the more general family of trochoid curves, as presented in his 1525 four-volume geometry treatise, The Art of Measurement with Compass and Straightedge, one of the first printed mathematical texts to appear in German. In this chapter, we characterize the hypocycloid geometrically. We then characterize it algebraically as a system of parametric equations, and dynamically as a differential equation. Finally we show that the hypocycloid is a solution to a minor variation of one of the most famous of mathematical riddles. But first we ask a natural question:

Did Drer use the trochoid in his woodcuts?

Drer argues at length that "geometry is that without which no one can either be or become a master artist" . . . . If he truly believed what he said, we have a measure of hope of finding abstract curves in his artwork.

MUST include links to the article "Black Holes through The Mirrour" from December, 2009 Mathematics Magazine. VERY cool. How to include as an investigation?

Important to note the link to Durer who first invented them. This should probably go in the introduction to the Spirograph, maybe with an image like:

Note that his drawings on perspective and his melancholia with the magic square give important links between mathematics and art as well!! Include them in the other relevant books.

CHAPTER 10

Interlude - Mathematics and the Visual Arts

> Mighty is geometry; joined with art, resistless.
>
> **Euripides** (; -)

> Geometry is the foundation of all painting.
>
> **Albrecht Durer** (; -)

Note the profound links to linear programming when we talk about the dissection of n-space by hyperplanes. Quote about the most important real world application. Full circle back to Art with Obaminoes.

> The linear-programming was – and is – perhaps the single most important real-life problem.
>
> **Keith Devlin** (; -)

Note the word PROBLEM!!!

Give links and put a picture in. Make note about Mathematics for Liberal Sciences. This is a good topic. But not enough people know about it or other contemporary mathematical developments and how they impact our life. I.e. plug MLS. Science and Art sybiosis.

Also have links to the fourth dimension in modern art. Talk some about cubism.

Links to perspective drawing and Annalisa Crannell's book. Talk about perspective and Brunelleschi?

Fractals!! Put in the Outkast Stankonia cd cover.

BIG note that all of this involves nontrivial mathematics. We're not talking a little bit of arithmetic. We are talking about fundamental changes to art. The use of perspective, fractals, basic shapes (Susan Sheridan and Ed Emberley), CAD and Photoshop, morphing and topology, computer graphics and programming (with modeling of creatures), fractals, photomosaics, cubism,... There has always been evidence that mathematics is involved in art (see the quotes above) - but now we are talking about deep, modern mathematics that students have likely not seen much of at all. And a great amount of this can be considered as patterns. It is absurd that students have not seen any of this.

Put the stuff in here about 3D printing? Or should that be in the sculpture book?

CHAPTER 11

Conclusion

This is a book. All books of any merit have some sort of a conclusion, a climax, a reminder of the main theme of the work. This should as well.

However, telling/stating/professing what this is is not compatible with the pedagogical approach of the book. We want to make it clear what the main theme/conclusion/climax is - but we also want to involve the students. So how do we structure a conclusion that is compatible with both of these needs?

Essay questions?

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