DISCOVERING THE ART OF MATHEMATICS

GAMES & PUZZLES

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

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with Julian F. Fleron and Philip K. Hotchkiss,

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Preface: Notes to the Explorer

Yes, that's you - you're the explorer.

"Explorer?"

Yes, explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a traditional book. Indeed, this book cannot be read in the traditional sense. For this book is really a guide. It is a map. It is a route of trail markers along a path through part of the world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore this path - to take a surprising, exciting, and beautiful journey along a meandering path through a mathematical continent named the infinite. And this is a vast continent, not just one fixed, singular locale.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by dozens of exercises that closely mimic illustrative examples. Rather, after a brief introduction to the chapter, the majority of each chapter is made up of Investigations. These investigations are interwoven with brief surveys, narratives, or introductions for context. But the Investigations form the heart of this book, your journey. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover the mathematics that is behind music and dance. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking, mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is discovered each day than in any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! *Fermat's Last Theorem*, which is considered in detail in Discovering that Art of Mathematics - Number Theory, was solved in 1993 after 350 years of intense struggle. The 1$ Million Poincaŕe conjecture, unanswered for over 100 years, was solved by **Grigori Perleman** (Russian mathematician; 1966 -). In the time period between when these words were written and when you read them it is quite likely that important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning mathematical skills, mathematical reasoning, and mathematical applications. Arithmetical and statistical skills are useful skills everybody should possess. Who could argue with learning to reason? And we are all aware, to some degree or another, how mathematics shapes our technological society. But there is something more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is one of its driving forces. As the famous **Henri Poincaŕe** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as both a liberal art and as a science. As a powerful science, mathematics shapes our technological society and serves as an indispensable tool and language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in many other fine, accessible books (e.g. [COM] and [TaAr]). Instead, our purpose here is to journey down a path that values mathematics from its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was rescued and preserved in monasteries. Knowledge was categorized into the classical liberal arts and mathematics made up several of the seven categories.[1] During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts - except in the contemporary classrooms and textbooks where the focus of attention has shifted solely to the training of qualified mathematical scientists. If you are a student of the liberal arts or if you simply want to study mathematics for its own sake, you should feel more at home on this exploration than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more. In your exploration here you will see that mathematics is a human endeavor with its own rich history of human struggle and accomplishment. You will see many of the other arts in non-trivial roles: dance and music to name two. There is also a fair share of philosophy and history. Students in the humanities and social sciences, you should feel at home here too.

Mathematics is broad, dynamic, and connected to every area of study in one way or another. There are places in mathematics for those in all areas of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

It is my hope that your discoveries and explorations along this path through the infinite will help you glimpse some of this beauty. And I hope they will help you appreciate Russell's claim that:

> ...The true spirit of delight, the exaltation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Finally, it is my hope that these discoveries and explorations enable you to make mathematics a real part of your lifelong educational journey. For, in Russell's words once again:

> ...What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have served as beacons to people who have explored the continents of mathematics throughout history.

[1] These were divided into two components: the *quadrivium* (arithmetic, music, geometry, and astronomy) and the *trivium* (grammar, logic, and rhetoric); which were united into all of knowledge by philosophy.

Navigating This Book

Before you begin, it will be helpful for us to briefly describe the set-up and conventions that are used throughout this book.

As noted in the Preface, the fundamental part of this book is the Investigations. They are the sequence of problems that will help guide you on your active exploration of mathematics. In each chapter the investigations are numbered sequentially. You may work on these investigation cooperatively in groups, they may often be part of homework, selected investigations may be solved by your teacher for the purposes of illustration, or any of these and other combinations depending on how your teacher decides to structure your learning experiences.

If you are stuck on an investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us: "If thee is no struggle, there is no progress." Keep thinking about it, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Investigation numbers are bolded to help you identify the relationship between them.

Independent investigations are so-called to point out that the task is more significant than the typical investigations. They may require more involved mathematical investigation, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The Connections sections are meant to provide illustrations of the important connections between mathematics and other fields - especially the liberal arts. Whether you complete a few of the connections of your choice, all of the connections in each section, or are asked to find your own connections is up to your teacher. But we hope that these connections will help you see how rich mathematics' connections are to the liberal arts, the fine arts, culture, and the human experience.

Further investigations, when included are meant to continue the investigations of the area in question to a higher level. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory.

Within each book in this series the chapters are chosen sequentially so there is a dominant theme and direction to the book. However, it is often the case that chapters can be used independently of one another - both within a given book and among books in the series. So you may find your teacher choosing chapters from a number of different books - and even including "chapters" of their own that they have created to craft a coherent course for you. More information on chapter dependence within single books is available online.

Certain conventions are quite important to note. Because of the central role of proof in mathematics, definitions are essential. But different contexts suggest different degrees of formality. In our text we use the following conventions regarding definitions:

- An *undefined term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An ***informal definition*** is italicized and bold faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.
- A **formal definition** is bolded the first time it is used. This is a formal definition that suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a biographical name appears it is bolded and basic biographical information is included parenthetically to provide some historical, cultural, and human connections.

Introduction: Games and Puzzles

There are lots and lots of different types of games.

There are team sports - football, baseball, soccer, lacrosse, hockey, etc., where each competition is called a game. Individual competitions are also often referred to as games - a game of pool for example - even if the games are often referred to as matches - as in tennis. Indeed, one of the biggest international gatherings involves games - the Olympic Games.

Now, for those who love sports and want to be part of them vicariously, we have fantasy sports. Here players pick fantasy teams and excel when their fantasy players excel.

There are board games: Chess, Checkers, Clue, Mancala, Blokus, Tic-Tac-Toe, 9 Man Morris, Battleship, Risk, Monopoly, Chutes and Ladders, Hi Ho Cherrio, Dungeons and Dragons, and Go. Of course, there are card games: Pitch (aka Setback), Poker of hundreds of sorts, Spit, Go Fish, and Pinochle. There are dice games. As we shall see, there are games that have been played by cultures all over the world with many types of tokens, counters, dice, and rules.

There are childhood games: hide and seek, king of the mountain, chase, tag, hopscotch, and cat's cradle. There are the classic video games: Pac Man, Galaga, Super Mario, etc. Of course, these were predated by the famous pinball - which is also a game of sorts.

There are the new generation of videogames which place us in a virtual reality: Wii sports, etc.

There are spinning top games, Hungry Hippo games, folded paper football games,

Games abound.

1. Make a list of the specific games that you liked best as a child. Explain, in general terms, what it was about these games that made you like them the best.

2. Make a list of the specific games that you like best now. Explain, in general terms, what is is about these games that make you like them best.

3. Do you see much, if any, mathematics being involved in these games? Explain.

As seen above, there's a potential for our discussion of games to become overwhelming. If we define games too broadly, it is unlikely we will be able to analyze things very effectively. E.g. there is not much that we can say about baseball by analyzing Setback and Pac Man. So we'd like to narrow things down a bit so when we consider games we can make progress studying them. So when we talk about games - one of the main topics of this book - from a mathematical perspective, what do we mean?

From this point forward when we say game, we will mean a mathematical game. What is that? We'll just provide an informal definition - a ***mathematical game*** is a multiplayer game whose outcome involves only strategy, logic, and chance. We will not consider sports where physical movement is essential. We will not consider typical videogames where hand-eye coordination is essential. Nor will we consider fantasy sports where the outcome is determined by others success in non-mathematical games.

This is not to say that mathematics plays no role in these types of games. Baseball players, coaches, owners, and fantasy owners are nearly fanatical statisticians. Many mathematical tools have

been brought to bear on games that we would not consider mathematical. But this is the topic for another book - one in applied mathematics.

4. Repeat Investigation **1** and Investigation **2**, focusing now on what we have agreed to call mathematical games.

So we have, at least tacitly, agreed on a definition for games. What about puzzles?

5. What do you think differentiates a puzzle from a game?

6. From this point forward we'd like to use the word puzzle to mean ***mathematical puzzle***. Give some examples of puzzles that you do not consider mathematical and some that you would consider mathematical in the spirit of our discussion about games. Explain your reasoning.

CHAPTER 1

Rubik's Cube

> The Cube is an imitation of life itself –or even an improvement on life. The problems of puzzles are very near the problems of life, our whole life is solving puzzles. If you are hungry, you have to find something to eat. But everyday problems are very mixed–they're not clear. The Cube's problem depends just on you. You can solve it independently. But to find happiness in life, you're not independent. That's the only big difference.
>
> **Ernő Rubik** (Architect; 1944 -)

1. History and background

The Rubik's Cube was invented by Hungarian architect Ernő Rubik in 1974 (see Figure 1). His original motivation for developing the cube was actually a curiosity about architectural configurations in space.

> I love playing, I admit it, I particularly love games where the partner, the real opponent is nature itself, with its really particular but decipherable mysteries. The most exciting game for me is the space game, the search of possible space shapes, that is to say the logical and concrete building of various layouts.

Known as the "Magic Cube," the Rubik's Cube became a hit in Hungary starting in 1977. The other person essential in making the Rubik's Cube famous was hobby mathematician and businessman Tibor Laczi. He was instrumental in bringing the cube to the west in the early 1979's. As a result, Rubik became the first self-made millionaire in the Communist block. As of January 2009, about 350 million Rubik's Cubes have been sold worldwide.

FIGURE 1. Ernő Rubik with a cube.

1.1. Random Moves. For those tempted to just keep making random moves in the hope of solving the cube, there are 43,252,003,274,489,856,000 different configurations of the cube, which is approximately forty-three quintillion.

Example Investigation: *Find a way to make concrete and visible how large a variety of forty-three quintillon different cube configurations are, and what that might mean for solving the cube by making random moves.*

As an illustration how to think about such an investigation, we include a few ways of reasoning and documenting our thinking right here.

To help us imagine how large forty-three quintillion is, we could put all these different cubes right next to each other in a long line, for example. How long do you think that line would be? Take a wild guess! Once around the island of Manhattan? From New York to San Francisco? Once around the globe? All the way to the moon? All the way to the sun? All the way to the next star? All the way to the next galaxy? All the way across the universe?

We could also wonder how long it would take us to cycle through all these different configurations if we could make one move with our cube every second (which is reasonable but doesn't give us much time for thinking). Take a wild guess!

Now, let us take a closer look together: Given that a cube is about 57 millimeters wide, the line of cubes would stretch for 261 light years[1]. For comparison, the sun is about 8 light minutes away, and the closest star to earth is about four light-years away[2]. Our own galaxy, the Milky Way, is about 100,000 light years across. The closest galaxy similar in size to our own, the Andromeda Galaxy, is about 2 million light years away. So the line of cubes would stay within our own galaxy, but extend far beyond the closest star. In fact, Spica–the brightest star in Virgo–is about 260 light years away; see Figure 2.

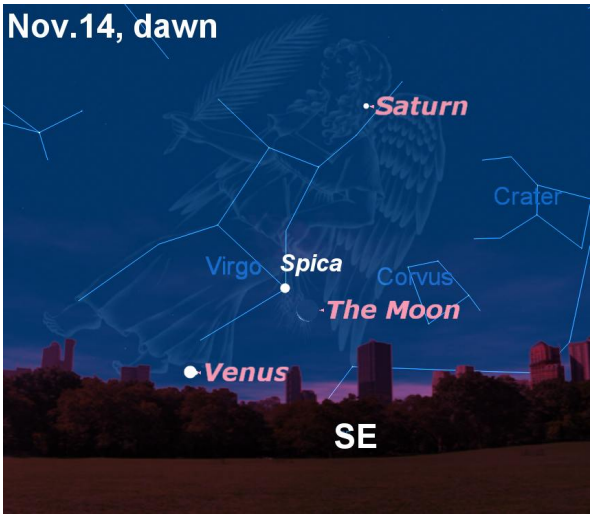


FIGURE 2. Virgo as seen on Nov 14, 2009: `http://www.thenightskyguy.com/`

We can look at this question in some other way: If it takes you one second to make one random move, and you manage to get a new configuration after every move, then it would take you at least 1.37 trillion years to make all the different configurations (at 31,557,600 seconds in a year). Scientists

[1]One light year: 9,460,730,472,580.8 km. Width of the cube: 57 mm.

[2]Proxima Centauri

estimate the age of the universe is about 13.75 billion years (give or take about 170 million years). Since 1.37 trillion is about 1370 billion years, it would take one hundred times the age of the universe to cycle through all possible configurations of the Rubik's Cube in this way. You may be lucky if the solved cube comes early—or you may end up twisting it in vain through the end of the universe.

2. Getting to Know the Cube

The Rubik's Cube consists of many little ***cubies***. The standard cube has six different colors: white, red, blue, green, orange, and yellow. The images we use have the following configuration: when completely solved, green is opposite yellow, orange is opposite red, white is opposite blue; as in `http://www.schubart.net/rc/`. If your cube looks different, don't worry!

1. How many cubies does the Cube have?

2. How many different stickers does each cubie have? Do they all have the same number of colored stickers? Explain in detail.

3. Pick a corner cubie. Can you move the cubie so that it's not in a corner any more? Explain your observations.

4. Pick a cubie that is not at a corner. Can you move this cubie into a corner position?

5. Turn your cube so that the white ***center cubie*** is facing up and the red center cubie is in front, facing you, as in Figure 3(a). Notice the marked corner cubie in the upper front left corner (the face that would otherwise be invisible on the left hand side is marked on the separately drawn face; imagine a mirror in this location). Clearly mark in Figure 3(a) all the different positions that you can get this corner cubie into, while keeping the white center cubie on top and the red center cubie in front. Explain any patterns you see.

6. Similarly, consider the marked ***edge cubie*** in Figure 3(b). Clearly mark in Figure 3(b) all the different positions that you can get this edge cubie into, while keeping the white center cubie on top and the red center cubie in front. Explain any patterns you see.

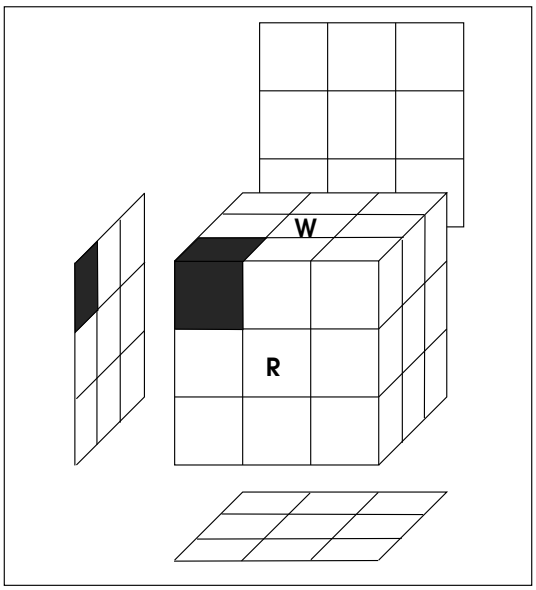
7. Finally, consider the marked center cubie in Figure 4. Clearly mark in Figure 4 all the different positions that you can get this center cubie into, while keeping the white center cubie on top and the red center cubie in front. Explain any patterns you see.

8. Summarize your findings and your reasoning in a few paragraphs so that you could present them to the whole class.

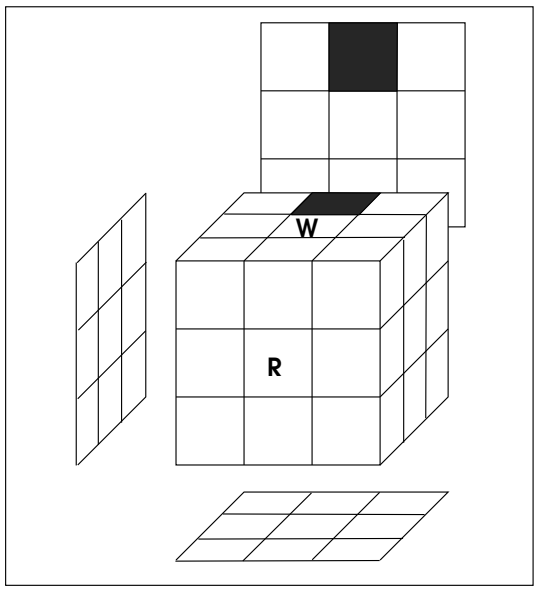
2.1. Solving One Face. The time needed for the following Independent Investigations may be measured in days, not in minutes.

9. Independent Investigation: Solve the white face on your own. It may be easiest to make white the top face, so you see what is going on. Take your time to get to know your cube. Observe which moves affect what cubies (and what cubies are left unaffected). You can do it!

Once you have one face all white, congratulations! The nine cubies that all have one white sticker form the top ***layer***. Thus, a layer consists of cubies, while a face consists of stickers. Now look at the color of the stickers around the edge of the white layer cubies. Do the three cubies next to each other on each face have the same color or different colors?



(a) Where can the shaded corner cubie go, while keeping the white and red centers in place?



(b) Where can the shaded edge cubie go?

FIGURE 3.

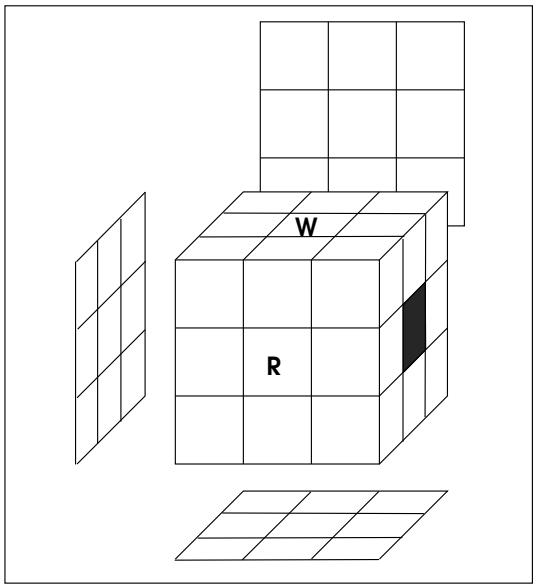


FIGURE 4. Where can the shaded center cubie go?

Consider, for example, Figure 5.

10. In Figure 5, we see a cube with a white side on top. Consider the four upper edge cubies (those with white stickers on top). Are they already in the correct position? How do you know? If not, describe where they would need to go in the completely solved cube (while keeping the white center cubie on top and the green center cubie in front).

11. Now, consider the four upper **corner** cubies. Are they already in the correct position? How do you know? If not, describe where they would need to go in the completely solved cube (while keeping the white center cubie on top and the green center cubie in front).

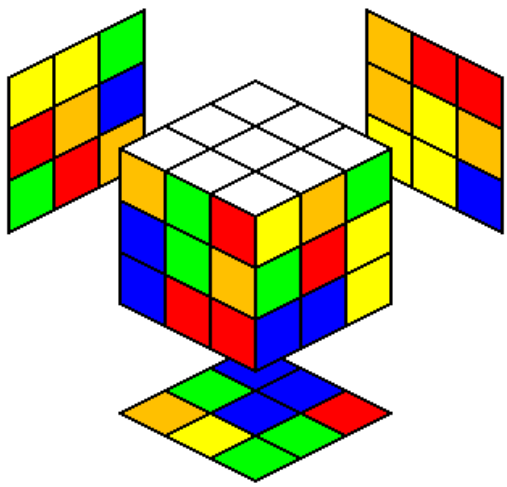


FIGURE 5. Solving the first layer.

12. Classroom Discussion: You may have noticed that it is difficult to precisely describe your strategies. What are effective ways we can use to clearly describe individual cubies, in words or in symbols? Explain why the methods you propose are clear and effective.

13. INDEPENDENT INVESTIGATION: Rearrange the top layer cubies so that the stickers along the edge on each face all have the same color. In the process, find your own methods for moving particular cubies in a way that leaves other cubies unaffected.

Once you know how to solve the white top layer, move on to next set of investigations, starting with Investigation **14**. They ask you to describe your moves for bringing certain cubies into a new position, while keeping the rest of the white top layer unaffected (i.e. once your moves are done, each of those cubies needs to be back in the place where it started out). We don't care at this point whether cubies in the second or third layer change position. It's OK if they end up in a different place. Find a way to describe your moves clearly enough so that somebody else could use these as instructions without any further aid.

3. Moving Specific Cubies in Specific Ways

The conventional notation for the different faces, usually named **Singmaster notation**, is shown in Figure 6: **U**p, **D**own, **F**ront, **B**ack, **L**eft, and **R**ight. Individual cubies are referred to using lowercase letters: for example, the corner cubie in the right, down, and back layer is called *rdb*, for short.

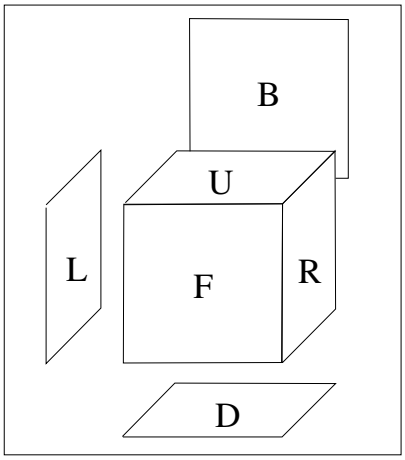


FIGURE 6. Naming the faces of the cube.

14. In Figure 7(a), complete the white top face; i.e. move the drf corner cubie into the urf position so that the white sticker faces up. Leave the rest of top layer unaffected.

15. Classroom Discussion: You may have noticed that it is hard to precisely describe your moves. What are effective ways we can use to clearly describe cube **moves**, in graphical representations, in words or in symbols? Why are the methods you propose clear and effective?

The conventional notation for denoting cube moves uses the abbreviations for the faces, as seen in Figure 6. We can turn a face clockwise as we look at it (denoted as U , D , F , B , R , L) or in a counter-clockwise direction (denoted as U^{-1} , D^{-1} , F^{-1} , B^{-1} , R^{-1} , L^{-1}). Notice that U^{-1} undoes precisely what the move U does; because of this, we call U^{-1} "U ***inverse***."

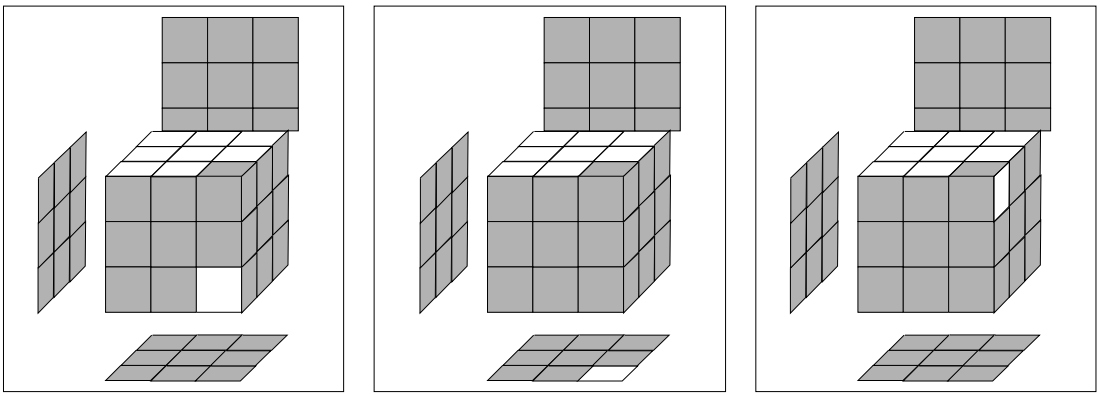
16. Complete the white top face in Figure 7(b) by moving the drf corner. Clearly describe your moves.

17. Complete the white top face in Figure 7(c) by fixing the urf corner. Clearly describe your moves.

18. Complete the white top face in Figure 8(a). Clearly describe your moves.

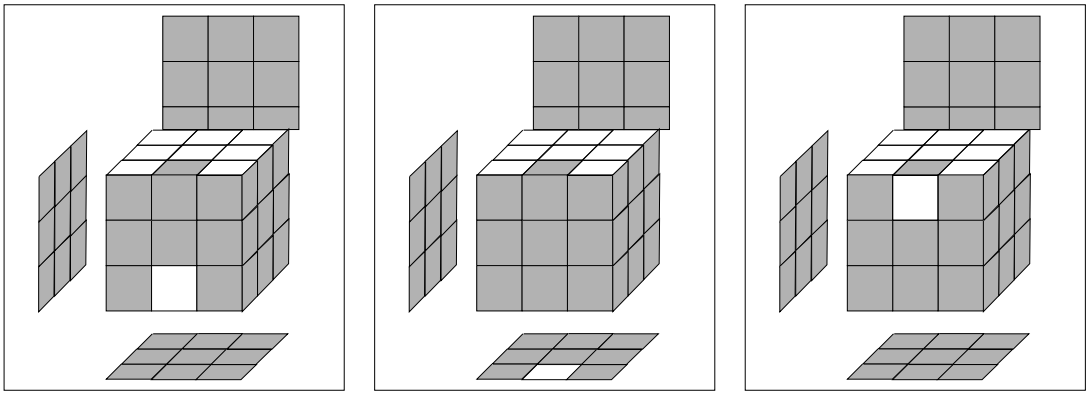
19. Complete the white top face in Figure 8(b). Clearly describe your moves.

20. Complete the white top face in Figure 8(c). Clearly describe your moves.



(a) Move the drf corner. (b) Move the drf corner. (c) Fix the urf corner.

FIGURE 7.



(a) Move the df edge. (b) Move the df edge. (c) Fix the uf edge.

FIGURE 8.

4. Magical Cube Moves

There are many ways of solving the cube. Some are very fast but may require you to memorize (or look up) many different specialized moves (see, for example, `www.speedcubing.com`). Other methods succeed with a smaller set of moves, but may take longer, or require you to start again from the very beginning when something goes wrong. Some methods start by solving the first layer (see Further Investigations 11). And, it has now been proven, that God can solve the cube, no matter what state it is in, in at most 20 moves.[3] Since we are mortal, we're going to begin slow by analyzing some very specific moves that are fairly magical. These will lead us, in the next section, to strategies for solving the cube.

The first magical move we will consider, which we'll label by M_1, is:

$$M_1 = R^{-1}DRD^{-1}.$$

The algebraic structure of this move certifies it as what is called a *commutator*. Indeed, the cube offers a wonderful *microworld* for the exploration of what is known as *modern abstract algebra*. The index of Adventures in Group Theory by **David Joyner** (; -) is a who's who of famous mathematicians and fundamental topics in modern abstract algebra - the body of the book suitable for an advanced undergraduate course for mathematics majors.

To begin experimenting with M_1, solve the top layer of the cube.

21. Beginning with the top layer of the cube solved, perform the move M_1. What impact did this move have on the top layer of the cube?

22. Perform M_1 again. Did it bring the top layer back? If not, what did it do?

23. Keep performing M_1 over and over again until the top layer is restored to its solved state. How many moves did it take you? Can you explain geometrically why it took you this number of moves?

We would like to understand not only what the move M_1 does to the cubies on the top layer of the cube, but to all cubies.

24. There are a number of cubies that are not moved by any of the moves that make up M_1. Describe exactly which cubies these are.

25. There are several cubies that are moved by the individual moves that make up M_1 but nonetheless, when M_1 is completed, these cubies return to their original location with their original orientation. Describe exactly which cubies these are. (We strongly suggest using a friend to help.)

26. There are several cubies who locations and orientations are changed by M_1. Describe the impact of M_1 on these cubies by:

- Drawing arrows on the cube in Figure 4 to show were moved cubies move to, and,
- Indicating precisely which cubies get interchanged with their names.

27. Now that you know the exact impact of M_1 on the cube, can you predict how many times in a row you will have to perform this move before the cube is brought back to the configuration you started from for the first time? Explain.

28. Check to see that your prediction is correct.

The number you found in Investigation **28** is called the ***order*** of M_1.

[3]The minimum number of moves needed to solve an arbitrary cube has been an open question, drawing a great deal of attention for many years. This number is known as **God's number**. The first positive result was that at least 18 moves were necessary and 52 moves suffice, proven by **Morwen Thistlethwaite** (; -) in 1981. In 1995 it was shown that the "superflip" required 20 moves, but the upper bound remained at 29 moves. It stood at this point for 10 years. The great breakthrough came in August, 2010. With the help of lots of computer CPU time at Google - Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge proved that 20 is both necessary and sufficient. See `http://www.cube20.org/` for more information.

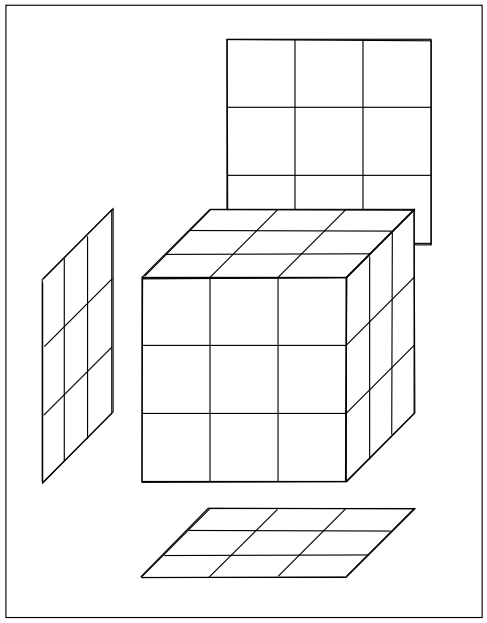


FIGURE 9. The effect of magic move M_1 on the cube.

Now we'll look at another magical move, which we will call M_2. It involves not only the usual moves, but a new move which you may have already used: M_R. M_R is a clockwise rotation of the middle slice parallel to the right side (clockwise as seen from the right side). Our new magic move is:

$$M_2 = M_R U M_R U M_R U M_R U.$$

Again, to begin, solve the top layer of the cube.

29. Can you think of a shorthand, algebraic way to write M_2?

30. Beginning with the top layer of the cube solved, perform the move M_2. What impact did this move have on the top layer of the cube?

31. Keep performing M_2 over and over again until the top layer is restored to its solved state for the first time. How many moves did it take you? Can you explain geometrically why it took you this number of moves?

We would like to understand not only what the move M_2 does to the cubies on the top layer of the cube, but to all cubies.

32. There are a number of cubies that are not moved by any of the moves that make up M_2. Describe exactly which cubies these are.

33. There are several cubies that are moved by the individual moves that make up M_2 but nonetheless, when M_2 is completed, the cubies return to their original location with their original orientation. Describe exactly which cubies these are. (We strongly suggest using a friend to help.)

34. There are several cubies who locations and orientations are changed by M_2. Describe the impact of M_2 on these cubies by:

- Drawing arrows on the cube in Figure 4 to show where moved cubies move to, and,
- Indicating precisely which cubies get interchanged with their names.

35. Now that you know the exact impact of M_2 on the cube, can you predict what the order of M_2 is? Explain.

36. Check to see that your prediction for the order of the move M_2 is correct.

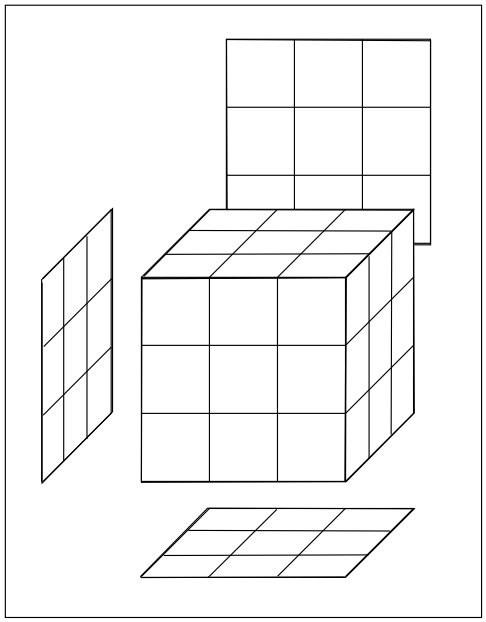


FIGURE 10. The effect of magic move M_2 on the cube.

37. We called M_1 and M_2 magical moves. We did that because they only move a small number of cubies. Why might this be useful when we are solving the cube?

38. If our goal is to find moves whose net results are to move a very small number of cubies, do you think the number of basic moves that are used in these moves are going to be small or large? Explain.

5. Solving the Cube: Corners First Style

As noted above, there are many different approaches to solving the cube. The main goal here is not just to solve one particular aspect of the cube **once** (e.g. having the top face show all white), but for you to develop reliable methods that will **always** allow you to solve certain aspects of the cube. These methods are related to the magical moves in the preceeding section.

Where do these magical moves come from? They come from great insight, research, deep geometric ideas, as well as insights from modern abstract algebra. They are not obvious. They may not seem natural. They are certainly not something that one could come up with easily. They are magical not just in what they do, but that they have been discovered at all.

There are several of these magical moves. Each will take a bit of time to understand and get used to. Once you have reached a particular goal, celebrate excitedly for a minute, congratulate yourself for excellent work, take careful notes on any important observations you made, draw a picture, take a photo, then mess up the cube and do it all over again. Once you've really got this move down, move on to the next.

This solution method is based on the book Adventures in Group Theory by David Joyner [**12**]. As before, the color of the center cubies tells us what color this face will be in the solved cube. Because of this fact, you can determine precisely which cubies should go where.

You may use moves from above. But be careful since they may scramble up parts of the cube that you have already solved.

The following method has the following main strategy:

- First bring all of the corner cubies into the correct position.
- Next bring all of the edge cubies into the correct position.
- Fix the orientations of the corner cubies.
- End by fixing the orientation of the edge cubies.

With this approach the cube may continue to look fairly messy for a fairly long time. It may be helpful for two people to work together on solving the cube this way.

5.1. Fixing the Positions of the Corner Cubies. Two tools - two new magical moves - are required to fix the eight corner cubies.

We will call the first move CR for **corner rotation**. It is:

$$CR = U^{-1}\ R^{-1}\ D^{-1}\ R\ U\ R^{-1}\ D\ R \tag{1}$$

This move effects exactly three cubies: ulf, urf, and drf.

39. By doing CR a number of times, determine the impact of this move on the ulf, urf, and drf cubies.
40. Draw arrows on the cube in Figure 11 to show where move CR moves cubies.
41. What is the order of the move CR?

We will call the next move CS for **corner swap**. It is:

$$CS = U\ F\ R\ U\ R^{-1}\ U^{-1}\ R\ U\ R^{-1}\ U^{-1}\ R\ U\ R^{-1}\ U^{-1}\ F^{-1} \tag{2}$$

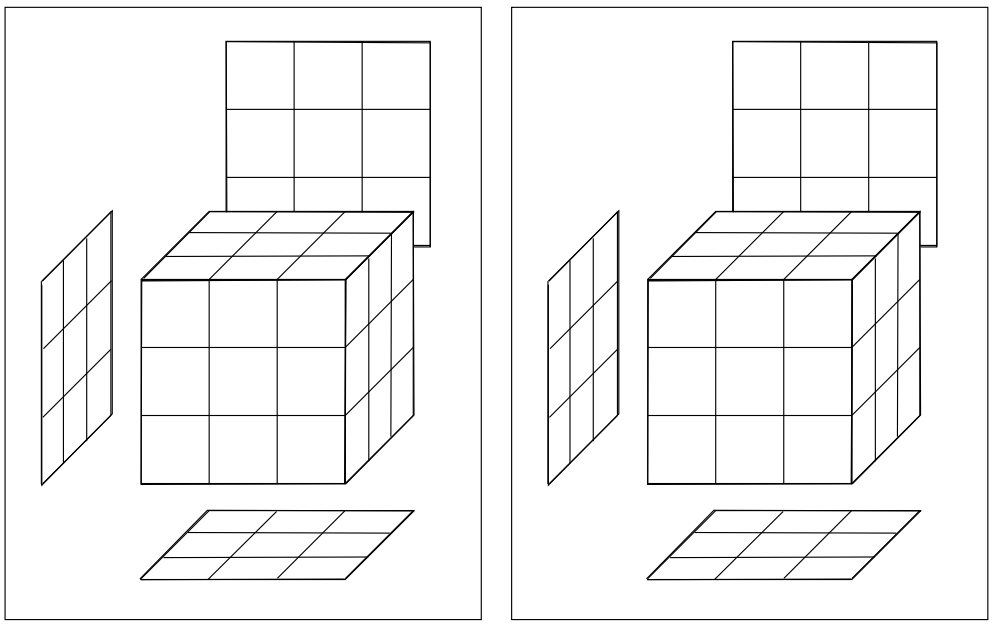
This move effects exactly eight up cubies - all of the up cubies but the center. No other cubies are effected by this move.

42. By doing CS a number of times, determine the impact of this move on the ubr and ufl cubies.
43. By doing CS a number of times, determine the impact of this move on the ubl and ufr cubies.
44. By doing CS a number of times, determine the impact of this move on the uf, ul, ub, and ur cubies.
45. Draw arrows on the cube in Figure 11 to show where move CS moves cubies.
46. What is the order of the move CS?

47. Solve a layer of your cube - a face with the adjoining row of cubies around the edges correct as well.

48. Now use the two moves you have learned in this section to try to fix the position of the remaining four corners of your cube without messing up your solved face. Were you able to? If so, celebrate! Then mix your cube up and do it again. If not, see if you can get some help from a peer.

49. Classroom Discussion: Come together with your peers and discuss the use of the moves CS and CR. Were each of you able to fix the four remaining corners? If some were not, can others help? If each cube's corners can be solved, can you explain how you know these two moves suffice? Alternatively, is there an example where you can describe precisely how these moves do not suffice?



(a) "The move CR"

(b) "The move CS"

FIGURE 11. Identifying the cubies affected by the moves described in (1) and (2).

5.2. Fixing the Positions of the Edge Cubies. Another move that is an important tool, which we call ER for **Edge Rotation**, is:

$$ER = M_R^2\, U^{-1}\, M_R^{-1}\, U^2\, M_R\, U^{-1}\, M_R^2. \tag{3}$$

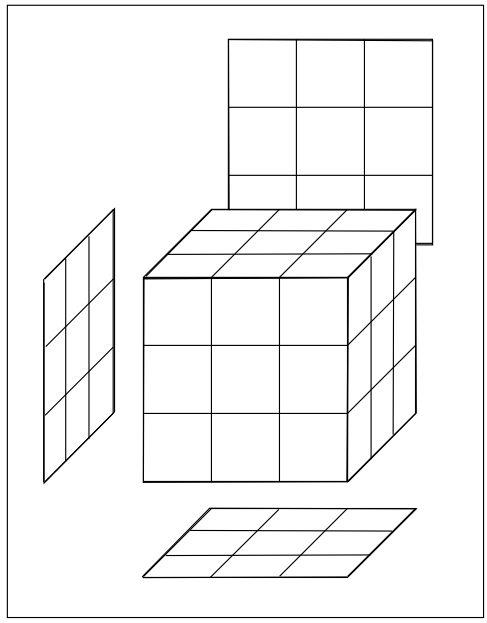
This move effects exactly three cubies: uf, ul, ur.

50. By doing ER a number of times, determine the impact of this move on the cube.

51. Draw arrows on the cube in Figure 12 to show where move ER moves cubies.

52. What is the order of the move ER?

53. Can you use this move to fix the position of *all* of the remaining edge cubies of your cube. Were you able to? If so, celebrate! Then mix your cube up and do it again. If not, see if you can get some help from a peer.



(a) "The move CR"

FIGURE 12. Identifying the cubies affected by the moves described in (3).

54. Classroom Discussion: Come together with your peers and discuss the use of the moves ER. Were each of you able to fix the edges? If some were not, can others help? If each cube's edges can be solved, can you explain how you know this move suffices? Alternatively, is there an example where you can describe precisely how this move does not suffice?

5.3. Fixing the Orientations. Now that you have all of the cubies in place, it is likely that many of them will not be oriented correctly. You've already seen that that move M_2 simply changes the orientation of cubes. You'll need this move in addition to two others that will help orient some of the other cubies that need to be reoriented.

We call the first CO for **Corner Orientation**. It is:

$$CO = (R^{-1}\ D^2\ R\ B^{-1}\ U^2\ B)^2. \tag{4}$$

This move effects exactly two cubies: ufr and dbl.

55. By doing CO a number of times, determine the impact of this move on the cube.

56. Draw arrows on the cube in Figure 13 to show how CO reorients cubies.

57. What is the order of the move CO?

58. Can you use this move to reorient *all* of the corner cubies of your cube? If so, celebrate! Then mix your cube up and do it again. If not, see if you can get some help from a peer.

We call the second EO for **Edge Orientation**. It is:

$$EO = (M_R\ U)^3\ U\ (M_R^{-1}\ U)^3\ U\ . \tag{5}$$

This move effects exactly four cubies: uf and ub.

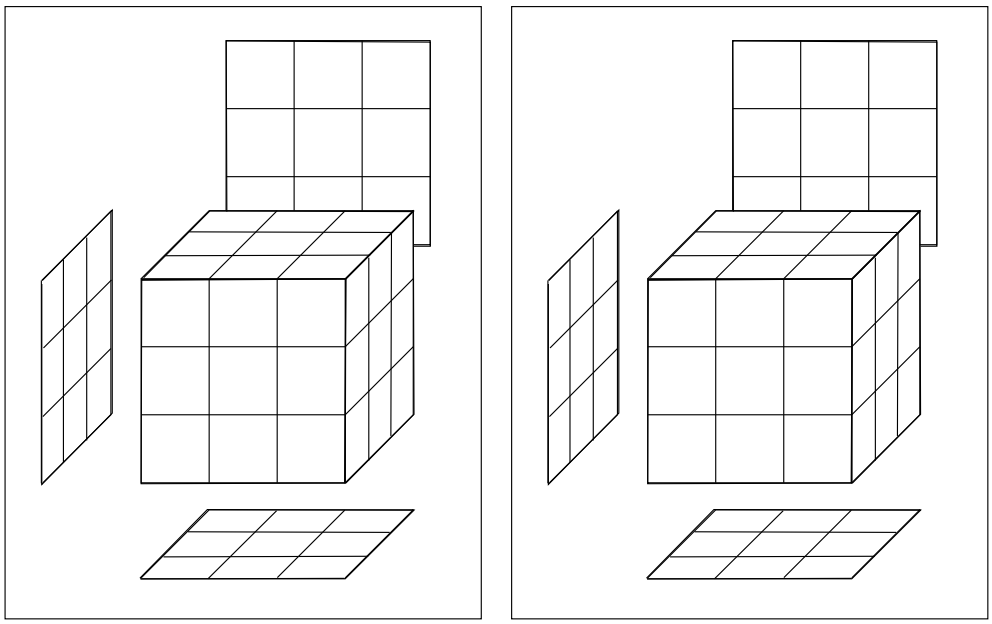
59. By doing EO a number of times, determine the impact of this move on the ub, ul, df, and db cubies.

60. Draw arrows on the cube in Figure 13 to show how EO reorients cubies.

61. What is the order of the move EO?

62. Use EO and M_2 to try to fix the position of the remaining four corners of your cube. These moves will suffice - you can convince yourself later. Get help from peers if needed. Then ... *Celebrate*!!

63. Classroom Discussion: What were some of the challenges you encountered in working only with the limited tools given in the Investigations? How did you manage to overcome them?



(a) "The move CO" (b) "The move EO"

FIGURE 13. Identifying the cubies affected by the moves described in (4) and (5).

6. Speed-cubing and Competitions

The first Rubik's Cube competition took place in Hungary in 1982. Since then, many different competitions have been taking place all over the world; see `www.speedcubing.com` for recent and upcoming events. In addition to "regular" speed-cubing, there are also "blindfolded" and "one-handed" competitions, as well as competitions where you can only use your feet (see Figure 14).

As of July 2010, the current world record for a single solve of the Rubik's Cube is 7.08 seconds, set by Eric Akkersdijk at the Czech Open on July 12-13, 2008. Go and check out some videos on YouTube.

64. Writing Assignment: Do some research on your own about cube competitions and speed-cubing. Write about your impressions. Did something surprise you? Would you be interested in speed-cubing?

FIGURE 14. Speed-cubing in Estonian Open 2009. Anssi Vanhala on his way to a new world record for solving the Rubik's Cube with his feet: 36.72 sec.

7. Analyze the moves

7.1. One Move, Two Moves, Three Moves. Guess!

Rules: This is a game for two players who take turns. Player One makes up a secret list of three turns (e.g. $R^{-1}\ D\ L$), takes the solved cube, and applies these three turns (again in secret). Hand the cube over to Player Two, clearly indicating the colors of the top and front faces. Player Two takes the twisted cube and now has to determine the exact sequence of turns. If Player Two's answer is correct, Player Two wins. If not, Player One wins. Switch roles and continue until each player had three attempts in each role.

To make it more challenging, use four or five secret turns.

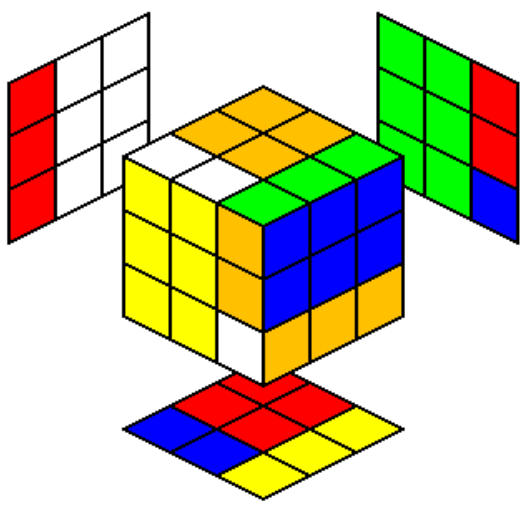


FIGURE 15. Two secret turns were applied to the solved cube.

65. In Figure 15, two secret turns were applied to a solved cube. Explain in detail what those moves were, and what the cube looks like at each stage in between.

66. In Figure 16, three secret turns were applied to a solved cube. Explain in detail what those moves were, and what the cube looks like at each stage in between.

67. What relationship do you see between the secret turns and the turns needed to undo them? Explain.

68. Classroom Discussion: How did you know which layers to turn? List some clues. How can you be sure that your guess is the correct solution? What is the pattern in finding the secret turns when you know the moves to undo? How does this relate to *inverses*?

7.2. Inverses. In Investigation **50**, you saw that the move

$$M_R^2\, U^{-1}\, M_R^{-1}\, U^2\, M_R\, U^{-1}\, M_R^2 \tag{6}$$

rotates the three cubies uf, ul, ur in a clockwise direction. Sometimes it would be more convenient to rotate them in a *counterclockwise* direction.

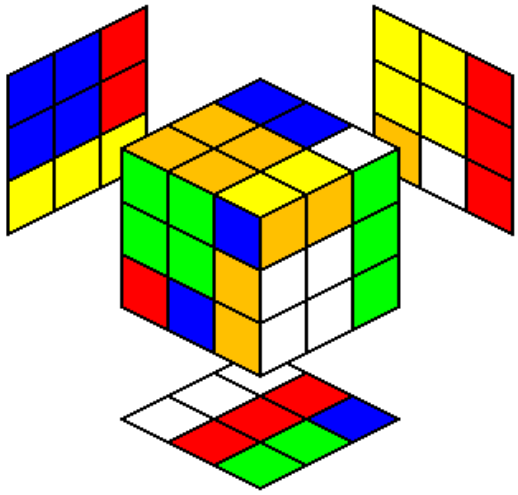


FIGURE 16. Three secret turns were applied to the solved cube.

69. Use our knowledge of the clockwise version in (6) to find a counterclockwise version. Explain your reasoning. *Hint: Recall Investigations* **65-67**

70. Mathematicians call moves such as (6) ***three-cycles***. Why do you think that is?

8. Patterns

8.1. Patterns.

71. INDEPENDENT INVESTIGATION: Starting with the solved cube, find a way to create patterns on your cube. Keep track of the moves you make to get to these patterns. Take photographs.

72. INDEPENDENT INVESTIGATION: Using your cube patterns from the previous investigation, imagine and design ways to put several cubes together to make a larger patterned design. Take photographs.

8.2. Multi-cube art. If we put several cubes together, we can create larger three-dimensional patterns, such as shown at `http://cube.misto.cz/hana.html`. Explore some designs of your own.

9. Further Investigations: Cubes with 4 and 5 layers

73. Now that you have been successful in solving the $3 \times 3 \times 3$ cube, you can challenge yourself by investigating one of the other size cubes. Explore how your strategies for the $3 \times 3 \times 3$ cube apply to solving the larger cubes, such as the $4 \times 4 \times 4$ and $5 \times 5 \times 5$ cubes, and what additional strategies you might need.

10. Further Investigations: Anatomy of Higher-dimensional Cubes

Figure 17 shows a view of a four-dimensional Magic Cube applet hosted at `http://www.superliminal.com/cube/applet.html`. Where your regular three-dimensional $3 \times 3 \times 3$ cube shows single-color, two-dimensional stickers, this four-dimensional $3 \times 3 \times 3 \times 3$ cube shows single-color, three-dimensional $3 \times 3 \times 3$ hyperstickers.

Just as with the regular Magic cube, you can apply turns to each of these. Try it out and see how your moves affect the various hyperstickers!

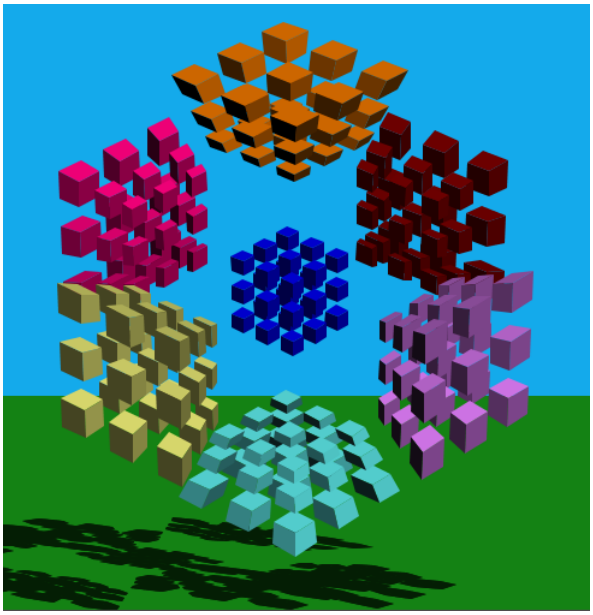


FIGURE 17. A four-dimensional Magic Cube, showing single-color $3 \times 3 \times 3$ cubes.

F1. How many 4-dimensional cubies does a 4-dimensional Magic Cube have?
F2. How many faces does a 4-dimensional (d-dimensional) Magic (Hyper-)Cube have? Explain.
F3. How many 3-dimensional "stickers" does a 4-dimensional Magic Cube have?
F4. What kinds of rotations are possible? Explain. Compare with the 3-dimensional Cube.

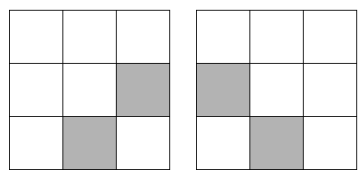
11. Further Investigations: Layer-by-Layer

This solution method is due to Singmaster. You already know how to solve the upper layer of the cube. The following investigations guide you through steps for the second and third layer.

11.1. Second Layer.

F5. Fix second layer(all the cubies underneath the solved upper layer): This describes how to get cubies from the bottom (third) layer into their appropriate place in the second layer. Look at the bottom layer. What is the color of its center cubie? Find a cubie in the bottom layer that does **not** have that color on any of its sides. This cubie really needs to go into the second layer (Why?).

What are the two colors of that cubie? Rotate the bottom layer so that the front color agrees with the center cubie of the side, forming a large T. This is your new front face. Check the other color of the edge cubie to decide whether it should go to the right (Figure 18(a)) or to the left (Figure 18(a)). The following moves will bring this edge cubie in place in the



(a) Edge cubie goes to the right. (b) Edge cubie goes to the left.

FIGURE 18.

second layer, while leaving the fixed upper layer unaffected.

To the right: $D^{-1}\ R^{-1}\ D^{-1}\ R\ D\ F\ D\ F^{-1}$

To the left: $D\ L\ D\ L^{-1}\ D^{-1}\ F^{-1}\ D^{-1}\ F$

Repeat until the entire second layer is fixed.

F6. Bottom Layer: Now turn the cube over so that the unsolved layer becomes the upper layer (so we can see what is going on).

F7. Get a cross-shape: In order to get a one-color cross, check which of the four pictures below best describes the state of the cross on your upper layer.

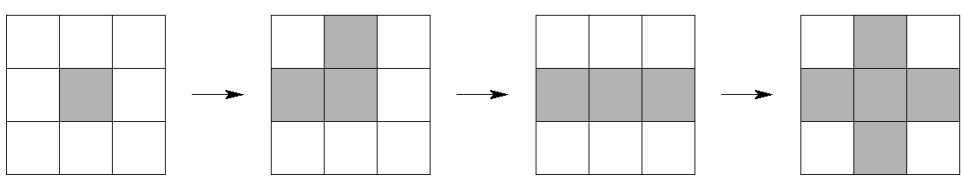


FIGURE 19. Steps to get a cross on top.

The sequence of moves

$$F\ R\ U\ R^{-1}\ U^{-1}\ F^{-1} \tag{7}$$

will take you along the arrows from one picture to the next until you get to get the cross.

F8. Correct edge cubies to match all the side-colors: Check if you can turn the top so that **exactly** one of the side colors matches. If you can, take that side as your front face and do the following moves:

$$R\,U\,R^{-1}\,U\,R\,U^{\,2}R^{-1} \tag{8}$$

You might have to do this move once or twice.

If there are always **two opposing** cubies with side colors matching, pick one as your front, do the above sequence of moves. Now you should be able to turn the top so that **exactly one** color matches. Go back up to 8.

You may encounter a case that does not match any of the two scenarios above. We leave this to you to figure out.

F9. Bring corner cubies into the correct position (that is, the corner cubie's stickers have the same colors as the three adjoining faces, but the stickers may be pointed the wrong way):

Check if there is **exactly one** corner cubie in the correct position. If so, turn the cube so that this cubie is in the top front right corner, and do the following sequence of moves:

$$U\,R\,U^{-1}\,L^{-1}\,U\,R^{-1}\,U^{-1}\,L \tag{9}$$

If **no corner cubie** has the correct colors for its position, then it doesnt matter what you pick as your front face. Do the move in 9 once, then check again whether you can now **exactly one corner** cubie matches. Then continue with 9 as above.

F10. Fix orientation of corner cubies: We now have all the cubies of the cube in the correct position, but some of the upper corner cubies may be facing the wrong way. We need to flip them. For this last step it is important to keep a cool head, since parts of the cube will get mixed up temporarily in the process.

Turn the cube so that the cubie you want to flip is in the upper right hand corner. Make a note of the color of the center cubie on the front face. It is **essential** that you keep this same center cubie in front until you are completely done with the cube. Do not lose your front center cubie.

Now start with these moves

$$R^{-1}\,D^{-1}\,R\,D \tag{10}$$

and repeat until your upper right hand cubie is fixed (i.e. all the colors face the right way).

At this point, other cubies of your cube may be mixed up. Dont worry, keep a cool head, take a deep breath, keep your front center cubie, and proceed to the next step.

If you need to flip other corner cubies, **keep your front center cubie**, but turn the top layer to bring that corner cubie into the upper right corner. Continue with the sequence of moves as shown in 10, until the very last corner fixes itself. At that point, you should see how finish the cube.

Congratulations! Enjoy your cube! Show off!

Then mess it up for practice and start the whole process all over again.

CHAPTER 2

Sudoku

1. Sudoku Rules

Regular Sudoku is a single-player puzzle consisting of a square 9×9 grid usually containing digits ranging from 1 through 9. A significant number of grid entries are empty. We call each of the digits that are given a **clue**. See Figures 1(a)-2(b) for a few example Sudoku puzzles. The large square is subdivided into nine 3×3 sub-squares. The goal is to fill the digits 1 through 9 into the open grid positions according to the following three rules:

- Each **row** must contain each of the digits 1 through 9 exactly once. (Notice that this means that each of those digits must occur somewhere along the row, and none of them can be duplicated.)
- Each **column** must contain each of the digits 1 through 9 exactly once.
- Each of the small 3×3 squares must contain each of the digits 1 through 9 exactly once.

Regular Sudoku puzzles are designed to have one unique solution, i.e. there is only one possible correct answer for each space in the grid. If you find yourself guessing between a variety of possible answers, you may guess incorrectly. Make sure you eliminate all but a single answer before filling in a final solution!

Sudoku puzzles are widely available in newspapers, books, and online. There is even an iPhone App.

Before we go any further, take a stab a solving a few puzzles. Figures 1-2 show four Sudoku puzzles of varying levels of difficulty: Figures 1(a)-1(b) are considered entry level, Figure 2(b) more challenging, with Figure 2(a) being somewhere in between.

1. Pick two of the puzzles in Figures 1-2 that are somewhat challenging for your level of experience with Sudoku. Solve these two puzzles. (If you are ready for an even more challenging Sudoku puzzles, you may replace these by more challenging ones. Clearly note your source.) As you solve the puzzle, keep an eye on the processes and the strategies you use.

2. Classroom Discussion: Summarize the processes and strategies you use to solve these puzzles and explain why they are helpful to you in solving the Sudoku puzzle.

3	2	4			9		7	
		7			2			8
8		6		5		1	2	
9	3		5	1				
			8		4			
				9	3		5	1
	4	3		7		8		9
6			9			4		
	8		6			5	3	7

(a) Entry level.

5					2		4	1
	4	1		8		3	6	
	6	3	9				8	
	5					6		
8		6		7		1		3
		9					7	
	3				8	2	9	
	9	5		2		8	1	
7	8		1					6

(b) Entry level.

FIGURE 1. Two Sudoku puzzles.

	3		5			7		
		6				9		
	4		9	2	7	1	6	
4		5		9	6			
			2		3			
			7	4		6		1
	5	2	8	7	9		1	
		4				5		
		3			4		9	

(a) Medium.

6			7			9		
								4
	2	9	1					6
		4	5		7			
9		5	4		6	2		3
			9		8	4		
2					1	3	9	
5								
		7			3			1

(b) More challenging.

FIGURE 2. Two Sudoku puzzles.

2. History

2.1. Latin Squares. Sudoku puzzles are an example of what the famous mathematician **Leonhard Euler** (Swiss mathematician and physicist; 1707 - 1783) called ***Latin Squares***: these are square tables filled with digits, letters, or symbols so that each of the entries occurs only once in each row or column. (Notice that Sudoku puzzles have the added requirement that each of the little 3×3 contain the digits 1–9 only once.) Figure 3 shows a beautiful example of a Latin Square which does not use digits or letters: in this stained glass window, each of the seven colors occurs only once in each row and in each column.

FIGURE 3. Displaying a Latin square, this stained glass window honors Ronald Fisher, whose "Design of Experiments" discussed Latin squares. Fisher's student, A. W. F. Edwards, designed this window for Caius College, Cambridge.

3. Working on your own, turn the color Latin Square displayed in the stained-glass window in Figure 3 into a Latin Square that uses numbers. Describe your process in detail. (If this printed document is not clear enough to make out the color detail, you will find a color image of the window at `http://en.wikipedia.org/wiki/File:Fisher-stainedglass-gonville-caius.jpg`.)

4. Now, find a completely different way of doing it. How do your results compare? Explain your observations.

5. Compare your results with those of other students. In what ways are the different Latin Square that were created the same? In what ways are they different? Explain.

6. How many different ways do you think there are to create a number version of the color Latin Square from Figure 3 (say, using the numbers $1-7$)? Document your process of thinking about this question. Carefully explain your answers and ideas.

Latin Squares were know long before Euler gave them this name, and in cultures far from the Latin world. You can find examples of Latin squares in Arabic literature over 700 years old. An ancient Chinese legend goes something like this: Some three thousand years ago, a great flood happened in China. In order to calm the vexed river god, the people made an offering to the river Lo, but he could not be appeased (Figure 4 shows a Ming dynasty scroll depicting the nymph of the Lo river). Each time they made an offering, a turtle would appear from the river. One day a boy noticed marks on the back of the turtle that seemed to represent the numbers 1 to 9. The numbers were arranged in such a way that each line added up to 15, as in Table 2.1. Hence the people understood that their offering was not the right amount.

Figure 4. Nymph of the Lo River, an ink drawing on a handscroll, Ming dynasty, 16th century. Freer Gallery of Art

4	9	2
3	5	7
8	1	6

Table 1. The Lo Shu magic square.

7. In the Lo Shu square, describe all the number patterns and relationships you can find.

Figure 5 shows one of the best-known depictions of a Latin Square in Western art. It is included in a woodcut called "Melencolia I" by the German artist **Albrecht Dürer** (painter, printmaker and theorist; 1471 - 1528), conventionally regarded as the greatest artist of the Northern Renaissance. Figure 6 shows "Melencolia" in its entirety. Dürer spent significant time in Italy and remained in communication with most of the major artists of the time, including **Raphael** (Italian painter; 1483 - 1520) and the proverbial Renaissance man **Leonardo da Vinci** (sculptor, painter, scientist; 1452 - 1519).

8. Transcribe the Latin Square from Dürer's woodcut Figure 5. Describe in detail as many patterns as you can find.

9. Writing Assignment: Research the history of Sudoku puzzles and write about your findings. Focus on two particular aspects that you find especially intriguing.

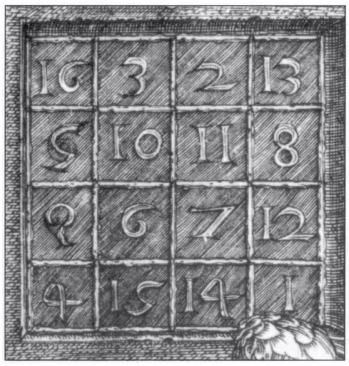


FIGURE 5. Latin Square from Dürer's woodcut *Melencolia*.

FIGURE 6. Albrecht Dürer created his famous woodcut *Melencolia* in 1514, showing a brooding figure which some believe is the prototype of Rodin's *The Thinker* and other renderings of people deep in thought. Notice the Latin Square to the right above the angel's head.

2.2. Sudoku. Source: Wikipedia,

- 18th century: Latin squares (Euler, Swiss)
- Late 1970's: Dell Magazines (US, "Number Place"), 1979.
- mid 1980 (1984): Nikoli (Japan, "Su: number", "Doku: single place"), 30 clues arranged symmetrically, Japan craze: 'Suuji wa dokushin ni kagiru' roughly translating to mean the numbers must be unmarried or single.
- Wayne Gould (NZ), Retired Hong Kong judge: computer program, "Su Doku"
- Nov 12, 2004: The Times (UK), "Su Doku"
- 2004: Wayne Gould, Conway Daily Sun (NH)
- 2005: worldwide craze
- 2008: "After 105 witnesses and three months of evidence, a drug trial costing $1 million was aborted yesterday when it emerged that jurors had been playing Sudoku since the trial's second week." [1]

[1] Knox, Malcolm (2008-06-11). "The game's up: jurors playing Sudoku abort trial". The Sydney Morning Herald.

3. Removing Clues

You may have noticed that the Sudoku puzzles in Figures 1-2 have different numbers of clues. How does this compare with the difficulty of these puzzles? One way we can try to make a puzzle more difficult (or at least require more work) is to remove some of the clues. (This is not the only way; see Section 4 for more information on creating puzzles.)

7			3		9	4	6	
4		1			2		8	9
6	3			8		2		
		4					1	
2	7						9	4
	1					7		
		7		2			3	6
1	6		5			9		7
	9	2	6		3			8

FIGURE 7. Remove one digit, making sure it still has a unique solution.

10. Classroom Discussion: In Figure 7, you can remove a "7" from the puzzle so that it still has a unique solution. Why can you be sure that the puzzle is still unique? Are there any other digits you can safely remove?

11. For the Sudoku puzzle in Figure 1(a), find one digit you could safely remove from the puzzle. (Make sure that it still has one unique solution!)

12. For the Sudoku puzzle in Figure 1(b), find one digit you could safely remove from the puzzle.

13. For the Sudoku puzzle in Figure 2(a), find one digit you could safely remove from the puzzle.

4. Creating Your Own Puzzles

4			
		1	
	2	3	
			4

(a) Joe's Sudoku puzzle.

		1	3
	2		
4			

(b) Karen's Sudoku puzzle.

FIGURE 8. Two 4×4 Sudoku puzzles.

14. Joe invented the Sudoku puzzle in Figure 8(a). What do you notice as you solve the puzzle?

15. Karen invented the Sudoku puzzle in Figure 8(b). What do you notice as you solve the puzzle?

16. It seems that creating Sudoku puzzles is not that easy. How do you think Joe and Karen should have gone about creating such puzzles?

17. Create four different puzzles yourself, then exchange them with a partner. Solve the puzzles you receive. Does each have a solution? Is it unique? (Figure 9 on page 36 has empty templates for 4×4 Sudoku puzzles.)

18. Classroom Discussion: From among the puzzles you created, pick your favorite one and share it with the class. Comparing all the puzzles, your creation strategies and the solutions, what patterns, similarities, and differences do you notice?

19. INDEPENDENT INVESTIGATION: It is an open problem to know what is the smallest number of clues needed for a 9×9 Sudoku puzzle[2] (with a unique solution). Investigate what is the smallest number of clues needed for a 4×4 puzzle (with a unique solution)?

20. INDEPENDENT INVESTIGATION: What is the **largest** number of clues you can have in a 4×4 puzzle that still has **no unique** solution?

For 9×9, the largest possible number of clues you can have without a unique solution is to have all but four squares filled with clues (81-4 = 77). Could you beat that for the 4×4 (i.e. have fewer than 4 open squares and still have no unique solution)? How about other size puzzles?

5. Sudoku Competition

21. Writing Assignment: What can you find out about Sudoku competitions around the world? What material did you come across in your research that you find most interesting. Write about your research.

6. 3D variants

22. Writing Assignment: How could we create a three-dimensional version of Sudoku? Write about your investigations (you can do research or create your own) and give examples.

7. Appendix: 4×4 Worksheets

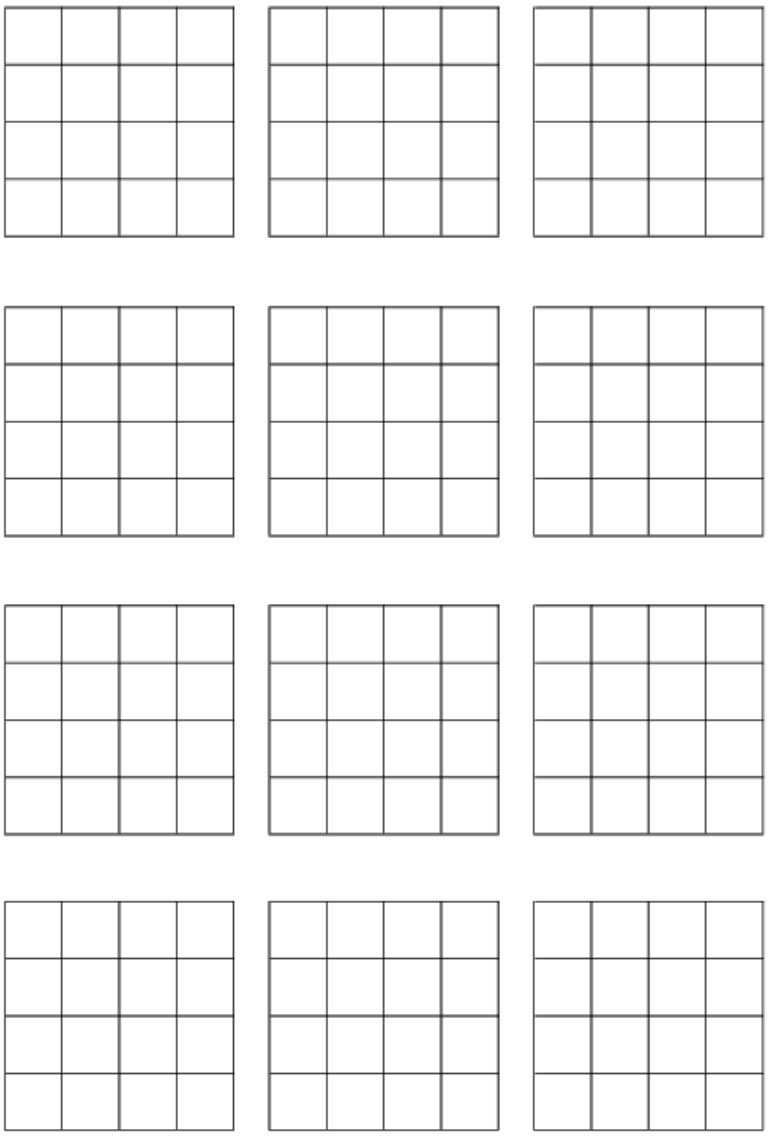


FIGURE 9. Templates for 4×4 Sudoku puzzles.

CHAPTER 3

Other Puzzles: Kakuro, Radon/Kaczmarz, and What's Inside of You

1. What made magic squares magical is that the sums of rows, columns, diagonals, and perhaps other groups of entries all have a common sum. What can you say about the sums of the columns of a Sodoku puzzle? The rows? The diagonals? Other groups of entries?

2. For a Latin square made with numbers, what can you say about the sums of the columns? The rows? The diagonals? Other groups of entries?

3. Use Investigation **1** to repharse the rules of Sodoku in terms of sums.

1. Kakuro

Kakuro is a puzzle with a history and popularity that resemble those of Sodoku. **Kakuro** "boards" are grids made of squares, looking much like a typical crossword puzzle board - many blank squares with a number of solid black squares. Taken together, all of the blank squares that are adjacent without interruption by a black square - in a given row or column are called a **run**. Like Sodoku, to solve a Kakuro puzzle you need to fill in all of the blank squares using only the numbers 1 - 9. The differences between the puzzles - other than their shape - are the rules and the clues:

- There is a clue for every run - the clue tells you what the ***sum of the terms in the run*** must be. No other clues are given; i.e. unlike Sodoku, no squares have been filled in.
- No number can be repeated in any run.

Like Sodoku puzzles, the solution to a Kakuro puzzle is expected to be unique.

4. Try to solve the Kakuro puzzle in Figure 1.

5. What did you find difficult about this puzzle? What did you find easy? What strategies did you use?

6. In Figure 2 is a *portion* of a Kakuro puzzle. How can you tell that it is only a portion of a puzzle?

7. Can you solve the portion of the puzzle in Figure 2? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

8. Similarly, can you solve the portion of the puzzle in Figure 3? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

9. Similarly, can you solve the portion of the puzzle in Figure 4? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

10. Similarly, can you solve the portion of the puzzle in Figure 5? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

11. Try to solve the Kakuro puzzle in Figure 6.

12. What did you find difficult about this puzzle? What did you find easy? Did you find any new strategies?

13. Can you solve the portion of the puzzle in Figure 7? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

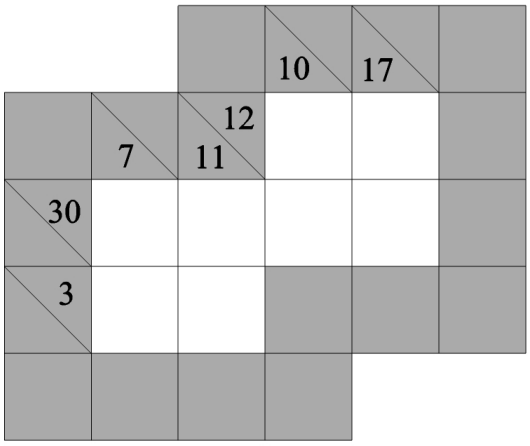


FIGURE 1. Junior Kakuro puzzle #1.

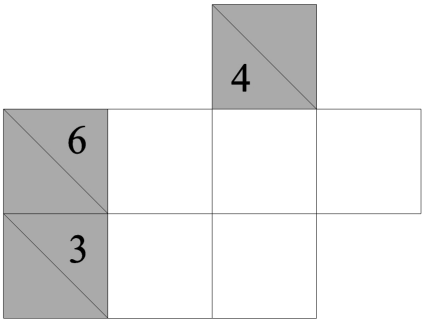


FIGURE 2. Kakuro excerpt #1.

By now you should be noticing that some clues are more valuable than other simply because the clue number can be broken up in only a few ways. For example, if our clue is a 7 for a run of three then the only possible combination is to use a 1, 2, and a 4. This is quite valuable. Frequent Kakuro players will have many of these examples at hand. For amateurs, like the authors, there are Kakuro Combination Charts available in many places on the Internet.

How many different possible combinations a number can take is actually an important mathematical topic. We call $7 = 1 + 2 + 4$ a ***partition*** of the number 7. There are 15 different partitions of the number 15. This includes many partitions that are not legal in Kokuro, since digits are repeated, like $7 = 2 + 2 + 1 + 1 + 1$. Partitions were first studied systematically by the great **Leonard Euler** (Swiss mathematician; -). Some of the most important progress was made by the great, but tragically short-lived **Srinivas Ramanujan** (Indian mathematician; -). For decades after Ramanujan's death the subject saw mainly minor advances. In 1999-2000, while studying Ramanujan's notebooks, **Ken Ono** (American mathematician; -) made a discovery that shocked the mathematical world - *partition congruences* must exist for every prime number, not just the small family they were thought to hold for. Shortly after his discovery that these partition congruences must exist, one of Ono's undergraduate students, **Rhiannon Weaver** (; -), found 70,000 partition congruences - mathematical patterns that were thought not to exist.

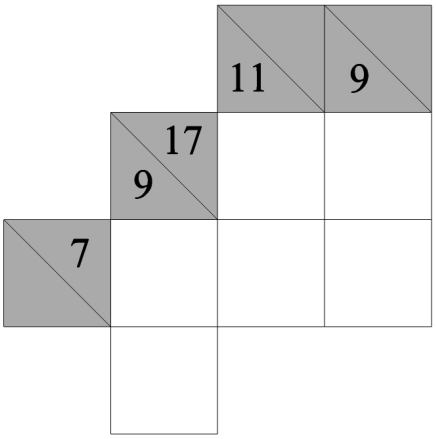


FIGURE 3. Kakuro excerpt #2.

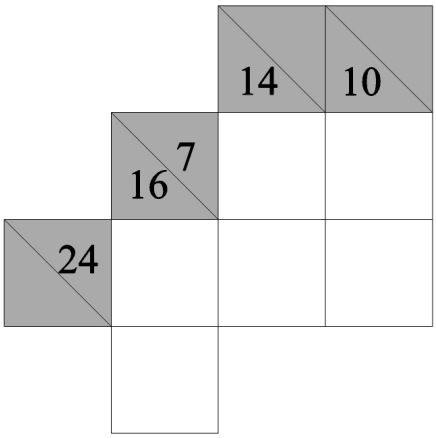


FIGURE 4. Kakuro excerpt #3.

Partitions make up the focus of two chapters of Discovering the Art of Number Theory in this series and the interested reader is encouraged to look there. Indeed, it was the discovery of Ono that motivated much of that book. And ironically, as this section of Kokuro was being written, Ono and several colleagues had another shock for the mathematical world. Partitions, counting the number of ways a given whole number can be written as the sum of other whole numbers, are inherently fractal in nature! Their discovery came in a remarkable fashion, as an Emory University press release tells us:

> A eureka moment happened in September, when Ono and Zach Kent were hiking to Tallulah Falls in northern Georgia. As they walked through the woods, noticing patterns in clumps of trees, Ono and Kent began thinking about what it would be like to walk amid partition numbers. "We were standing on some huge rocks, where we

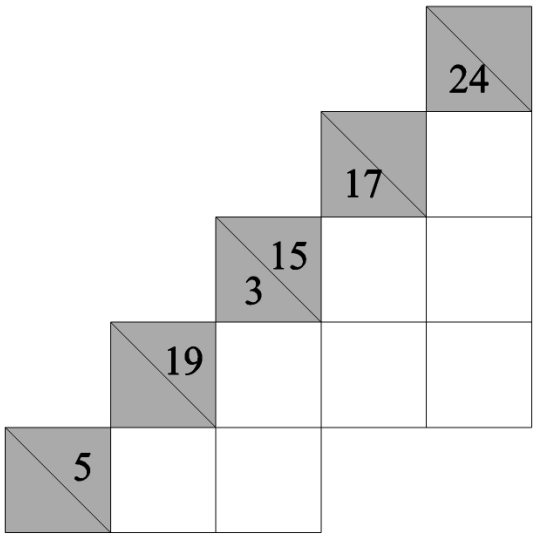


FIGURE 5. Kakuro excerpt #4.

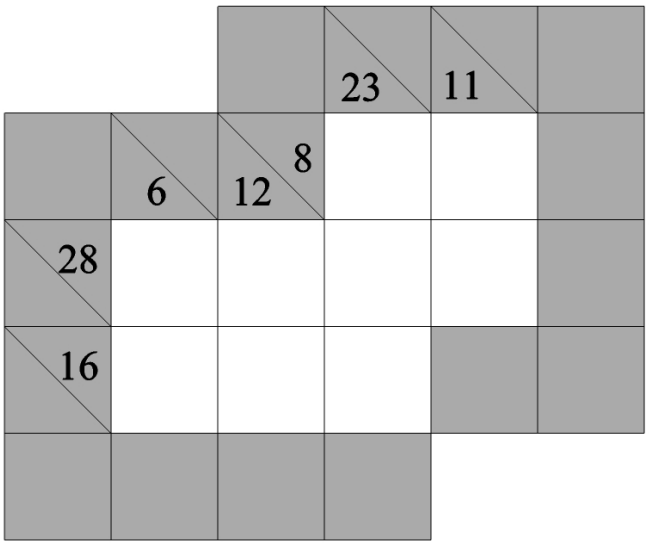


FIGURE 6. Junior Kakuro puzzle #2.

> could see out over this valley and hear the falls, when we realized partition numbers are fractal," Ono says. "We both just started laughing."[1]

Never underestimate the power of your subconscious.

[1] "New theories reveal the nature of numbers," by Carol King available at the `http://esciencecommons.blogspot.com/2011/01/new-theories-reveal-nature-of-numbers.html`.

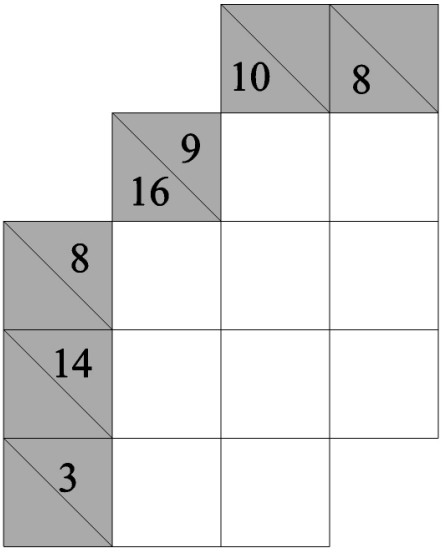


FIGURE 7. Kakuro excerpt #5.

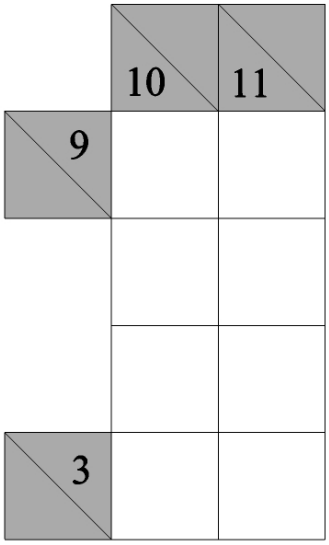


FIGURE 8. Kakuro excerpt #6.

14. Similarly, can you solve the portion of the puzzle in Figure 8? If so, can this portion of the puzzle be solved uniquely? Whatever your answer, prove that your result is correct.

15. There are many Kakuro puzzles available online - of all different levels of difficulty. Find one that you thinks is near the top end of your ability level. Solve this puzzle. Describe the major challenges.

2. Radon/Kaczmarz Puzzles

The next puzzles are our own invention and we have chosen to name them **Radon/Kaczmarz puzzles**, or **RK puzzles** for short, in honor of the mathematicians **Johann Radon** (Austrian mathematician; 1887 - 1956) and **Stefan Kaczmarz** (Polish mathematician; 1895 - 1940) whose work we will describe later.[2]

Like Sodoku and Kakuro puzzles, RK puzzles involve filling in grids with numbers constrained by certain rules and satisfying certain clues. In RK puzzles the numbers entered into the grid are required to be whole numbers 1 - 9, only now numbers *can* be repeated. The clues for RK puzzles are sums of terms, like in Kakuro, only here the sums take many different forms. We will refer to these sums as ***aggregates***. Aggregate data will be given visually, as in Figure 9 below. In this figure the aggregate data and puzzle grid are given on the left, the solved puzzle on the right.

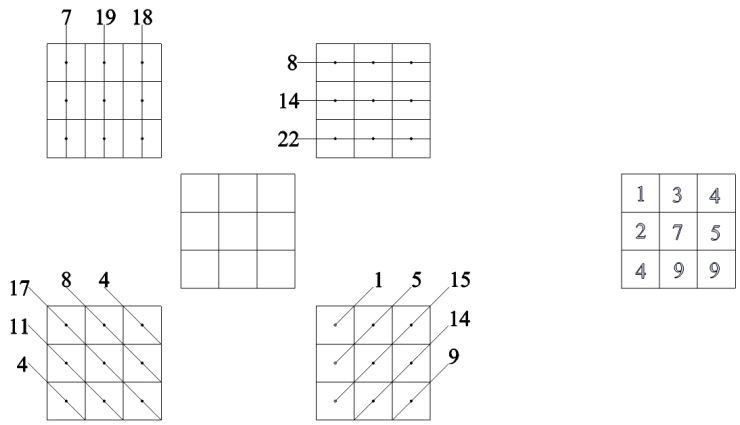


FIGURE 9. RK puzzle board and aggregate data (left) and solved puzzle (right).

16. Can you solve the RK puzzle #1 in Figure 10 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

17. Can you solve the RK puzzle #2 in Figure 11 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

18. Can you solve the RK puzzle #3 in Figure 12 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

19. Can you solve the RK puzzle #4 in Figure 13 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

20. Can you solve the RK puzzle #5 in Figure 14 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

21. Can you solve the RK puzzle #6 in Figure 15 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

22. Find simple conditions that are necessary for an RK puzzle to have a solution. These are conditions that can be checked without having to try to explicitly solve the puzzle. These conditions should be robust enough that they explain why any of the insolvable RK puzzles above are in fact not solvable. Prove that these conditions are necessary.

[2]After their development, we learned of the *Challenger puzzle* which is quite similar to a 4 by 4 R/K puzzle and which appears in a number of newspaper puzzle columns.

23. Are the conditions in Investigation **22** sufficient conditions for an RK puzzle to have a solution? I.e. if the conditions are satisfied does this guarantee that there is a solution to the RK puzzle? Prove that your result is correct.

24. Make and then prove a *positive result* about the solution of *arbitrary* 3 by 3 RK puzzles where the aggregate data includes the 3 vertical aggregates, the 3 horizontal aggregates, and both the 5 left and 5 right diagonal aggregates.

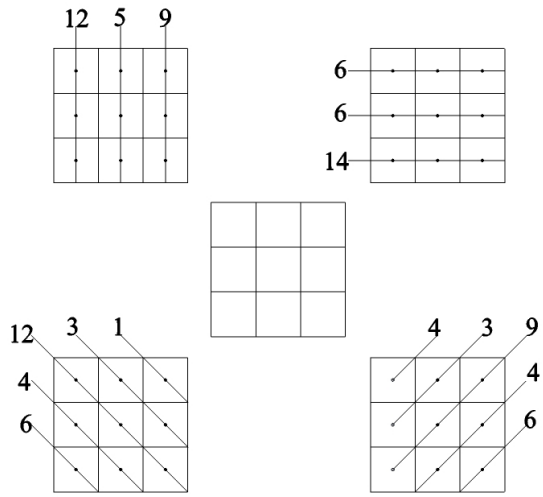


FIGURE 10. RK puzzle #1.

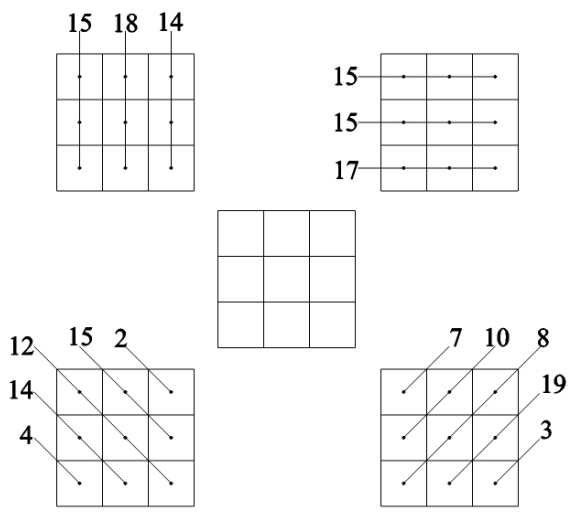


FIGURE 11. RK puzzle #2.

We are now going to move on to larger RK puzzles.

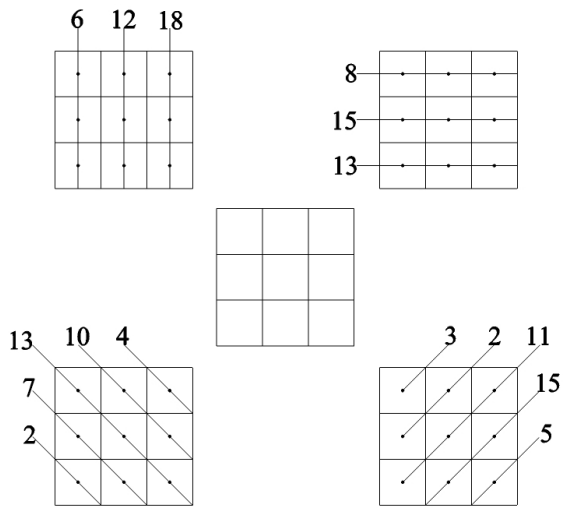


FIGURE 12. RK puzzle #3.

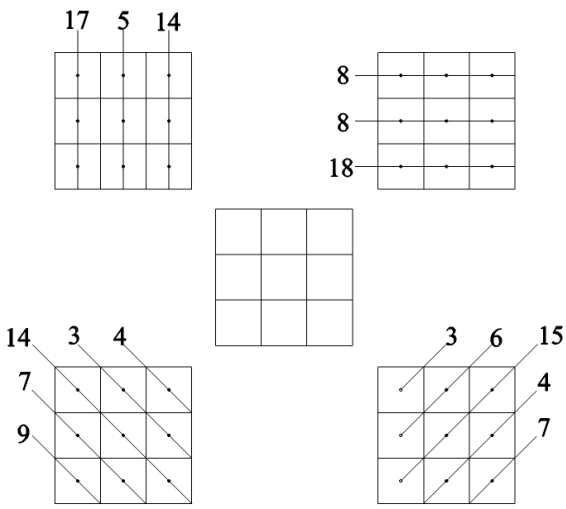


FIGURE 13. RK puzzle #4.

25. Can you adapt the conditions you found in Investigation **22** to 4 by 4 puzzles? If so, explain how. If not, explain why not. What about RK puzzles that are larger than 4 by 4? Explain.

26. Can you solve the RK puzzle #7 in Figure 16 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

27. Can you solve the RK puzzle #8 in Figure 17 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

28. Can you solve the RK puzzle #9 in Figure 18 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

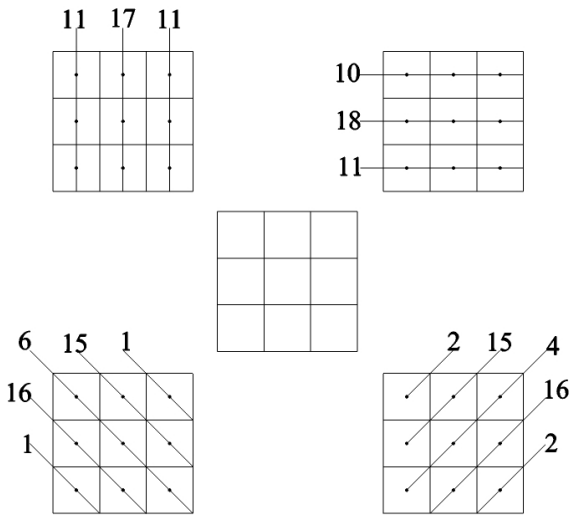


FIGURE 14. RK puzzle #5.

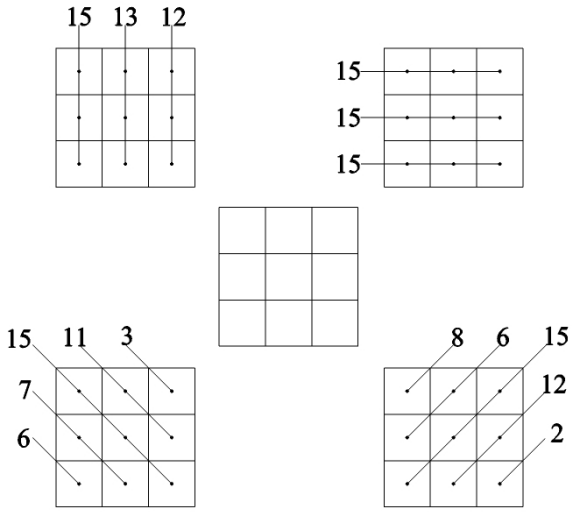


FIGURE 15. RK puzzle #6.

29. Can you solve the RK puzzle #10 in Figure 19 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

30. Can you solve the RK puzzle #11 in Figure 20 with the given aggregates? If so, is the solution unique? Whatever your answer, prove that your result is correct.

31. Suppose that you are given the additional ***shallow left diagonal*** aggregate data, shown in Figure 21, for RK puzzle #11 in Investigation **30**. Can you solve the puzzle? If so, is the solution unique? If so, do you need all of the additional data? Whatever your answer, prove that your result is correct.

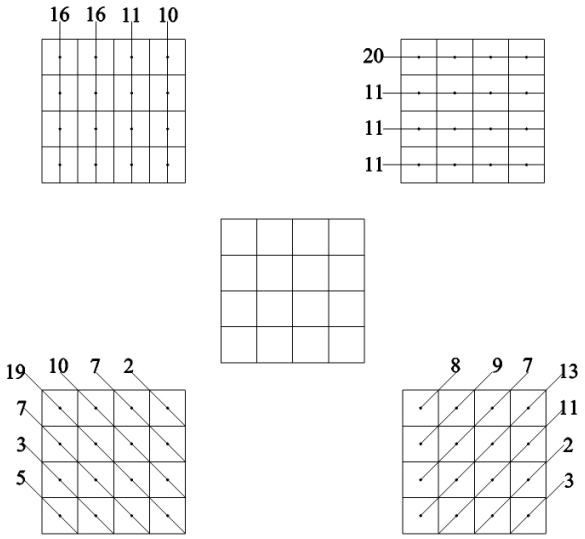


FIGURE 16. RK puzzle #7.

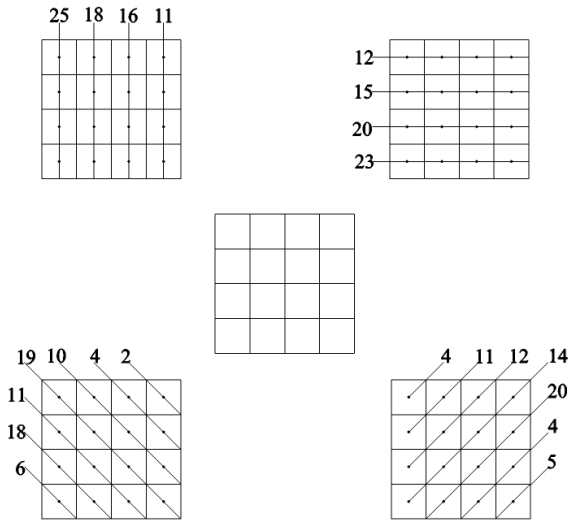


FIGURE 17. RK puzzle #8.

32. Make and then prove a positive result about the solution of *arbitrary* 4 by 4 RK puzzles where the aggregate data includes the 4 vertical aggregates, the 4 horizontal aggregates, both the 7 left and 7 right diagonal aggregates, and as few shallow left diagonal aggregates as possible.

33. Make your own 4 by 4 RK puzzle whose given aggregate data is: 4 vertical, 4 horizontal, 7 left diagonal, 7 right diagonal, and 10 shallow left diagonals. Explain carefully how you made the puzzle. (Note: Templates are included in the appendix.)

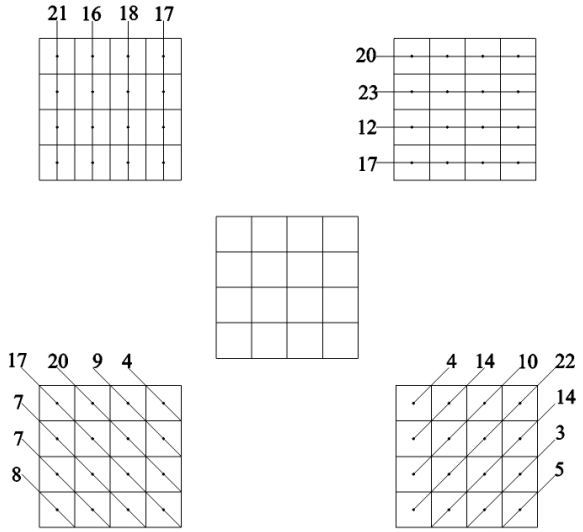


FIGURE 18. RK puzzle #9.

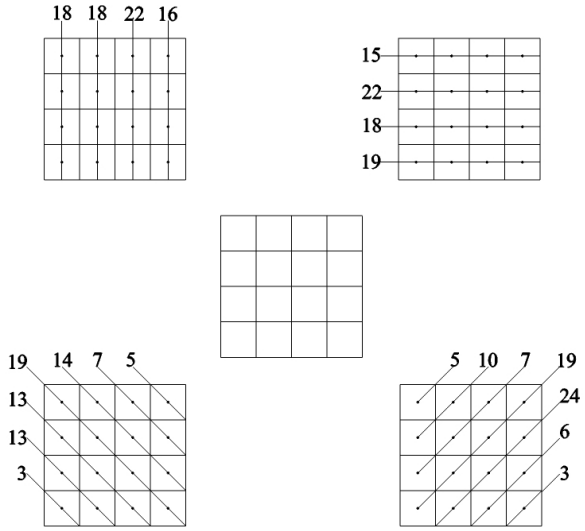


FIGURE 19. RK puzzle #10.

Sodoku, Kakuro, and RK puzzles all assume that there is a unique solution to be had - and one expects sufficient clues to be given or it is not a valid puzzle. In each, our job is to *reconstruct* the solution from incomplete data. Such reconstruction makes puzzles like this part of the important class of problems known as *inverse problems*.

34. Make your own 5 by 5 RK puzzle whose given aggregate data is: 5 vertical, 5 horizontal, 9 left diagonal, 9 right diagonal, and 13 shallow left diagonals. Explain carefully how you made the puzzle. (Note: Templates are included in the appendix.)

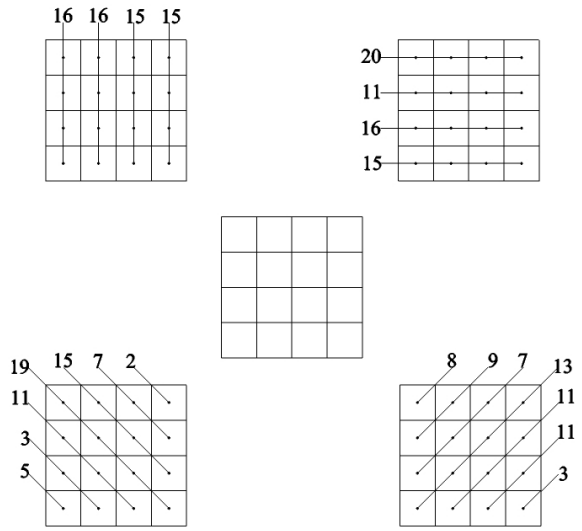


FIGURE 20. RK puzzle #11.

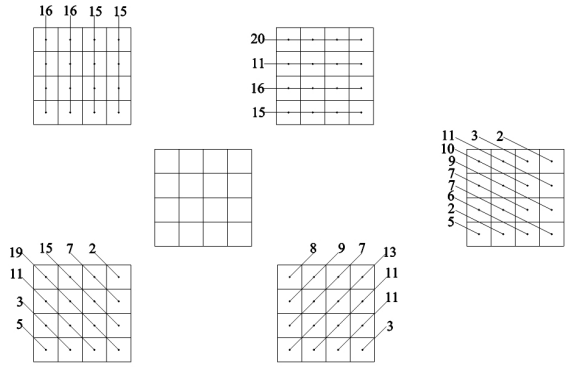


FIGURE 21. Shallow Left aggregates for RK puzzle #11.

35. Switch 5 by 5 RK puzzles with a partner - giving them only the aggregate data. Were you able to solve the puzzle they gave you with the given aggregate data? Were they?

36. Make and then prove a positive result about *arbitrary* 5 by 5 RK puzzles where the aggregate data includes the 5 vertical aggregates, the 5 horizontal aggregates, both the 9 left and 9 right diagonal aggregates, and the 13 shallow left diagonal aggregates.

In general, having 6 vertical, 6 horizontal, 11 left diagonal, 11 right diagonal, and 16 shallow left diagonal aggregates is not sufficient information to solve a 6 by 6 RK puzzle.

37. Given that such a 6 by 6 RK puzzle is not uniquely solvable, what other aggregates can you think of that might be useful as clues?

38. Can you solve the RK puzzle #12 in Figure 22 with the given aggregates? If so, solve it. If not, explain why it cannot be done.

39. Can you solve the RK puzzle #13 in Figure 23 with the given aggregates? If so, solve it. If not, explain why it cannot be done.

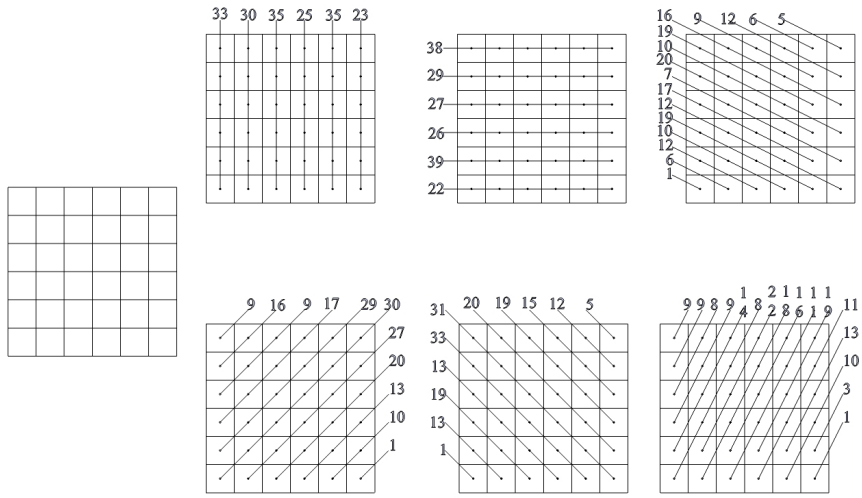


FIGURE 22. RK puzzle #12.

As the size of RK puzzles increase, it is interesting - and important - to know how many aggregates may be required to uniquely solve any solvable puzzle. This is a question that arises in algebra often - *systems of equations* that do not have enough data to be solved uniquely are called *under-determined systems* while those that have enough data to be solved uniquely are called *over-determined systems*.

40. Let's begin by considering 2 by 2 RK puzzles. Can you find the smallest number of aggregates that will insure that there is a unique solution to an *arbitrary* solvable puzzle of this size? Explain.

41. You have collected some data in your investigations that may help suggest how many aggregates are necessary. Complete the following table based on your investigations above. Here "Number of Variables" means that total number of pieces of data that must be recovered to solve the RK puzzle and "Number of Clues Needed" is an upper bound on the number of aggregates necessary to insure that a solvable puzzle has a unique solution.

Size of RK Puzzle	Number of Variables	Number of Clues Needed
2 by 2	4	
3 by 3	9	
4 by 4		
5 by 5		
6 by 6		
9 by 9		

42. For an n by n RK puzzle, how many pieces of data need to be recovered to solve the puzzle? Based on your table above, can you make a conjecture about the approximate number of aggregates that are needed to insure that there is a unique solution?[3] Explain.

[3]Notice the qualifying terms "conjecture" and "approximate." This is, as far as we know, an open problem. It may be a decent research problem for an advanced, undergraduate mathematics majors.

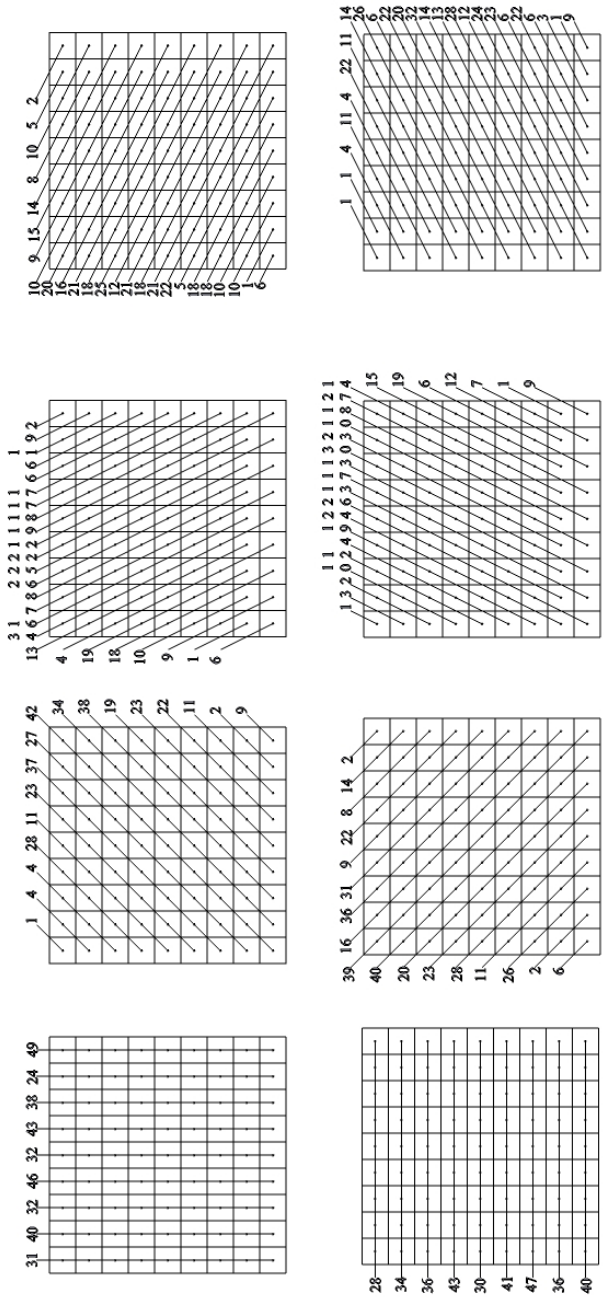


FIGURE 23. RK puzzle #13.

43. Roughly, how many aggregates do you think may be necessary to solve a 100 by 100 RK puzzle? Explain.

44. Roughly, how many aggregates do you think may be necessary to solve a 10,000 by 10,000 RK puzzle? Explain.

3. Why Are We Doing This?

You have explored lots of different puzzles of similar types. We hope that you have found them of interest; that they challenged your intellect. At this point you may be ready to move on to something new. Certainly a 100 by 100 RK puzzle does not seem like it would be much fun. A few years ago after working through much of this material and then tiring of RK puzzles an exasperated student said, "Why are we doing this? You said this was important. It better be a cure for cancer or something, because I'm about done." He wasn't far off.

Figure 24. Johann Radon, circa 1920.

In 1917 **Johann Radon** (Austrian Mathematician; 1887 - 1956) developed what has become known as the *Radon transform*. Developed to solve geometric problems he was studying, this transform is a cousin the *Fourier transform* which, among many other critical things, is a central tool in the conversion of digital data into analog forms; i.e. it is how your iPod plays music! While the Fourier transform plays a central role in mathematical analysis, the Radon transform was, for a long time, not very widely recognized.

Radon had a long and distinguished career in mathematics.

In 1937 **Stefan Kaczmarz** (Polish mathematician; 1895 - 1940) developed what has become known as the *Kaczmarz method* for algebraically solving systems of linear equations. Like the Radon transform, this development was not very widely heralded.

In September, 1939 both Nazi Germany and the Soviet Union invaded Poland. Poland was quickly overrun. It's military leaders and officers were all imprisoned. Also imprisoned were many of its police, doctors, public servants, lawyers, priests and academics. A huge number of these prisoners of war were eventually killed. Some 22,000 were killed in the ***Katyn Forest Massacre*** in the forests near Katyn, Russia. Mass graves were discovered by the Nazis in 1943. While the Nazis were quickly blamed, many believed the massacre was carried out by the Soviet Secret Police. In 1990, the Soviet Union finally took responsibility for this massacre. In 2010 **Vladmir Putin** (Russian Prime Minister; -) invited **Donald Tusk** (Polish Prime Minister; -) to a memorial to commemorate the 70th anniversary of the massacre. On 7 April, Tusk attended the memorial. In a horribly creul twist of fate, three days later a plane carrying **Lech Kaczyński** (Polish President; 1949 - 2010), his wife, and over 80 high ranking goverment and military officials to the memorial crashed, killing all on board. The location of the crash? Just outside the Katyn Forest.

Stefan Kaczmarz is likely one of those Poles who was murdered in the Katyn Forest Massacre. His known mathematical accomplishments are few.

Figure 25. Stefan Kaczmarz.

While the lives and careers of Radon and Kaczmarz were quite different, there share a critical historical relationship. Namely, the Radon transform and the Kaczmarz method are essential components of the development of CAT scans and other forms of medical imaging.

Ever had a CAT scan? An MRI? A PET scan? As early as 1975 it was suggested that this sort of medical imaging "may effect a revolution in medicine comparable to that brought about late in the

19th century by the introduction of anesthetics and sterile techniques.[4] It is safe to say that these forms of medical imaging are among the most important medical advances of the twentieth century. With the recent advent of *functional magnetic resonance imaging*, also known as fMRI, where we can watch pysiological changes in brain function in real time, the revolution continues still.

These medical imaging breakthroughs were made possible - in large part - by the Radon transform and the Kaczmarz method. These two mathematical methods are absolutely essential to the reconstruction of the images from collected data.

Remarkably, the work of Radon and Kaczmarz was so little known that these tools were reinvented as the need arose during the development of these medical imaging technologies! **Richard Gordon** (; -), **Bender** (; -), and **Gabor T. Herman** (; -) rediscovered Kaczmarz's method in 1970 and it is often called the *algebraic reconstruction technique*. **Allan M. Cormack** (South African doctor and physicist; 1924 - 1998) and **Godfrey Hounsfield** (English Engineer; 1919 - 2004) reinvented major aspects of the Radon transform.

So important were these discoveries that Cormack and Hounsfield shared the Nobel Prize for Medicine in 1979.

What would have happened if Radon and Kaczmarz work been remembered in this context? Gordon, Herman, and Johnson tell us "If Radon and the early tomographers had been aware of their common problems, many of the developments of the past few years might have been launched a half a century ago."[5] Remarkable!

Cormack's Nobel Lecture is beautiful and includes a wonderful reminder of the importance of intellectual engagement:

> What is the use of these results? The answer is that I don't know. They will almost certainly produce some theorems in the theory of partial differential equations, and some of them may find application in imaging with N.M.R. or ultrasound, but that is by no means certain. It is beside the point. Quinto and I are studying these topics because they are interesting in their own right as mathematical problems, and that is what science is all about.[6]

But what does all of this have to do with Puzzles and Games? The purpose of the Radon transform and the Kaczmarz method is to reconstruct data. They are highly technical approaches. The difficulties of practically utilizing them in a clinical setting are immense. Nonetheless, the way in which they enable us to create medical imaging technology like CAT scans shares a great deal of resemblence with the puzzles we have been solving. Honestly.

Tomography is from the Greek *tomos*, meaning slice, and *graphein*, meaning to write. It is used contemporarily to describe a type of imaging, often medical, where three dimensional objects are depicted by images that represent slices sectioned from the object - originally by means of an X-Ray, but now also by ultrasound, positron beams, sonar, or magnetic resonance.

An apple we can *cross section* with a knife, as shown in Figure 26. We can think of any three-dimensional solid as made up of its cross sections. Original *sliceforms* made by geometry students illustrate this idea in a nice way; see Figure 27. Of course, if we want to know what a living human being looks like inside - physically slicing is not "optimal." Instead, we let waves and beams do the slicing.

If you want more information on the relationship between three-dimensional objects and their two-dimensional cross sections, please see our Website "Navigating Between the Dimensions" at `http://`

[4] From "Image Reconstruction from Projections," by Gordon, Herman, and Johnson, *Scientific American*, vol. 233, no. 4, 1975, p. 56.

[5] Ibid.

[6] Allan M. Cormack from his Nobel Lecture, December 8, 1979.

`www.wsc.ma.edu/ecke/flatland/Home.html` or the chapters "Navigating Between the Dimensions" and "Dimensional Interplay in Other Fields" from Discovering the Art of Mathematics - Geometry.

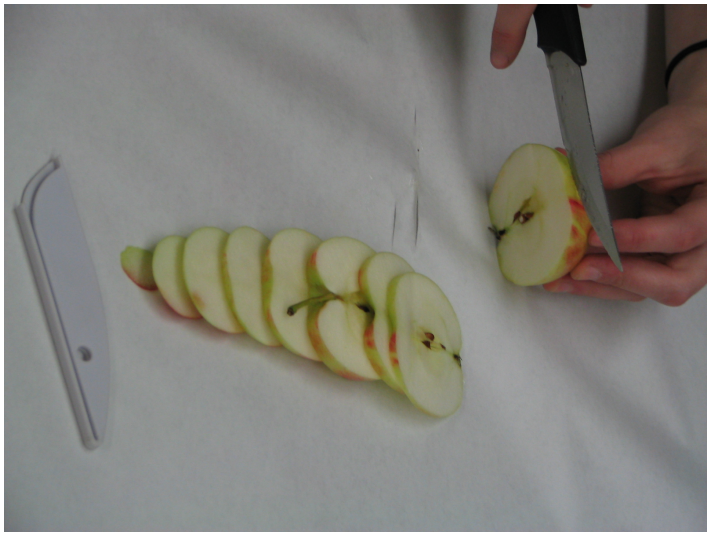


FIGURE 26. Cross sections of an apple.

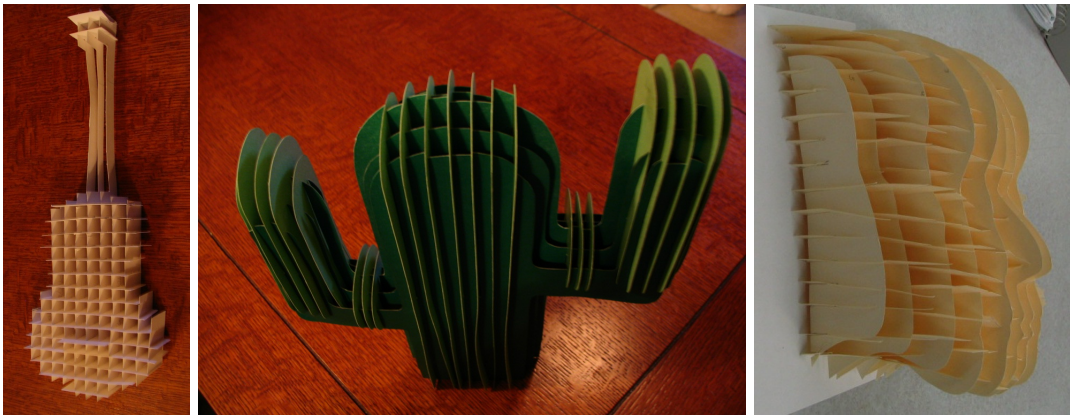


FIGURE 27. Original student sliceforms; "Guitar" by Katherine Cota, "Cactus" by Sharon Kubik-Boucher, and "The Face" the Lydia Lucia.

So how do beams of waves or particals help us slice without damaging what they are helping us image? We'll illustrate using a CAT scan - ***computerized axial tomography*** - one of the most widely used forms.

The functional "guts" of a CAT scan are shown in Figure 28. The long axis of your body extends through the hole in a perpendicular direction. This is the direction the slices are generally taken in humans - the ***axial direction*** - contributing the "A" in CAT scan. Measured beams of X-Rays are released as illustrated by the red arrows. Those that pass through the body are measured on the opposite side. As the beams and detectors spin, many thousands of X-Ray beams pass through the

axial plane of your body that is being imaged. These create one "slice." The detectors and beams then move up to take the next slice - perhaps a millimeter higher up. This happens with great speed so many thousands of slices are created.

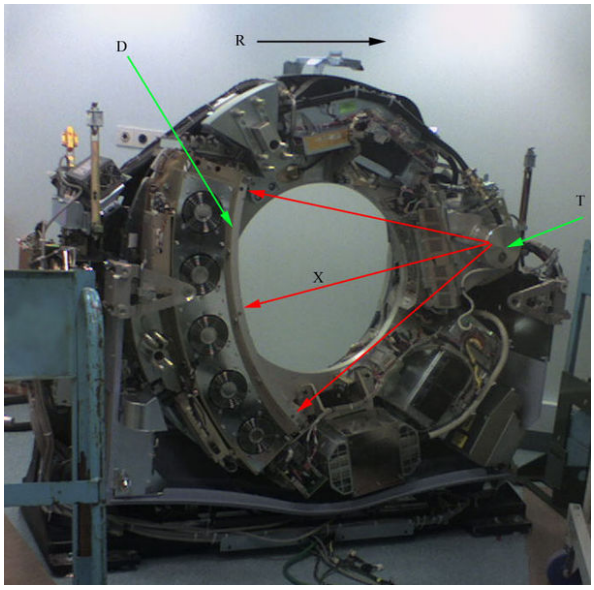


FIGURE 28. A modern (2006) CT scanner with the cover removed, demonstrating the principle of operation. Source: Wikimedia.

But how do the beams allow us to capture an image of the slice? Well, the intensity of the beams deminish as they pass through your body. They are ***attenuated***. How much they are attenuated depends on what they pass through. The *attenuation coefficients* are quite different for bone, soft tissue, muscle, tumor, etc. Imagine the axial cross section of your body pixellated by a fine grid, as illustrated by Figure 29. Each arrow indicates an X-Ray beam. And each number represents the amount of radiation that was attenuated along the path of the ray. If we could determine how much radiation was attenuated by each pixel then we would know - from its attenuation coefficient - what it was; bone, soft tissue, muscle, tumor, etc.

And, this is exactly the type of problem that we were trying to solve when we were working on RK puzzles! Indeed, the work of Radon and Kaczmarz - reinvented later because it had not been broadly enough known - is what allows us to recover the data needed to image the slice from the individual measurements of thousands of beams.

Of course, there are significant technical issues that make the use - and advancements - in this type of imaging quite complex. Among other things, think of the amount of data that would be required to solve an RK puzzle whose resolution is about 10,000 by 10,000 - the typical size of a decent image! Nonetheless, the underlying mathematical ideas at the heart of CAT scans are those ideas that you have discovered in solving RK puzzles. You understand the fundamental working components of critically important way of visualizing what is inside us - as illustrated by the remarkable images in Figure 30.

A wealth of information on imaging of this form is available online. Visible Human Program viewers available online. (E.g. the NPAC/OLDA Visible Human Viewer at `http://www.dhpc.adelaide.edu.au/projects/vishuman2/VisibleHuman.html` or the viewer at the Center for Human Simulation at `http://www.uchsc.edu/sm/chs/browse/browse.htm`.) The book Medicine's New Vision has

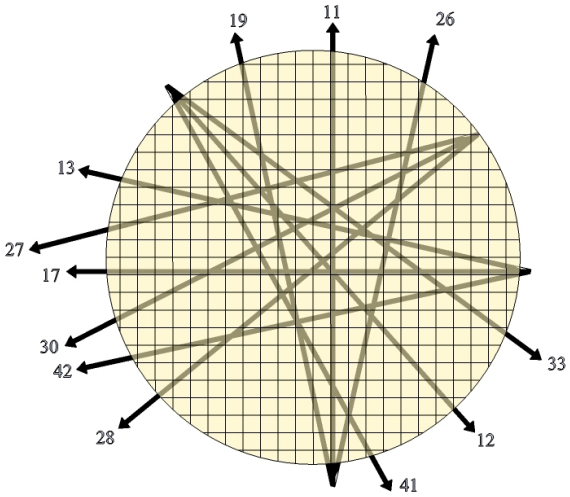


FIGURE 29. Illustration of a pixellated cross section bombarded with X-Ray beams and their attenuation coefficients.

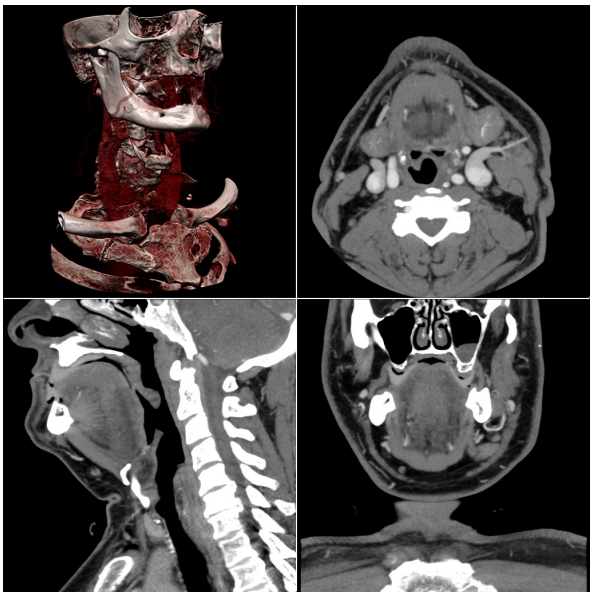


FIGURE 30. Typical screen layout of workstation software used for reviewing multi-detector CT studies. Clockwise from top-left: Volume rendering overview, axial slices, coronal slices, sagittal slices. Source: Wikimedia.

beautiful images and is quite readable. If you are interested in the hard-core mathematics required to actually put these ideas into place, The Mathematics of Computerized Tomography and Introduction to the Mathematics of Medical Imaging will quickly indicate what a substantial challenge this is.

CHAPTER 4

Connection Games – Hex, ConHex, ...

1. Hex

1.1. Rules of Hex and Playing Hex. Hex is a board game played on a hexagonal grid, theoretically of any size and several possible shapes, but traditionally as an 11x11 rhombus; see Figure 1. Other popular dimensions are 13x13 and 19x19 as a result of the game's relationship to the older game of Go. According to the book *A Beautiful Mind*, John Nash (one of the game's inventors) advocated 14x14 as the optimal size. John Nash won the Nobel prize for economics in 1994; see Figure 2.

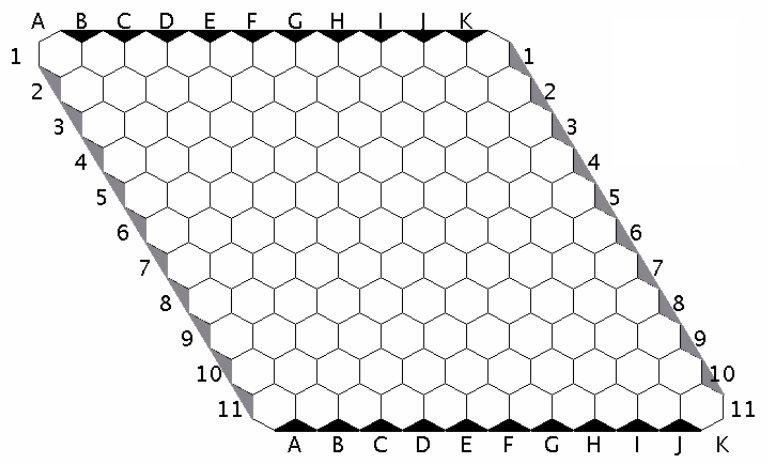


FIGURE 1. An 11x11 Hex board.

Each player has an allocated color, Red and Blue being conventional. In this book, we will use Black and Gray instead. Players take turns placing a stone of their color on a single cell within the overall playing board. The goal is to form a connected path of your stones linking the opposing sides of the board marked by your colors, before your opponent connects his or her sides in a similar fashion. The first player to complete his or her connection wins the game. Figure 3 shows an example of Gray forming a complete path connecting both sides thereby winning the game. The four corner hexagons each belong to two sides. As you place a stone, you don't have to place it adjacent to another of your color. Any open hexagon on the board can be chosen.

In class, we have opponents to play against. If you want to practice at home or explore strategies on your own, you can find Hex applets online[1] or download an application. For some of these, you can play against a computer or against somebody else on the net.

[1] The most popular sites are boardspace for real-time play, http://www.ludoteka.com/ for real-time play, http://games.wtanaka.com/hex/ for real-time or turn-based play, igGameCenter (http://www.iggamecenter.com/) for

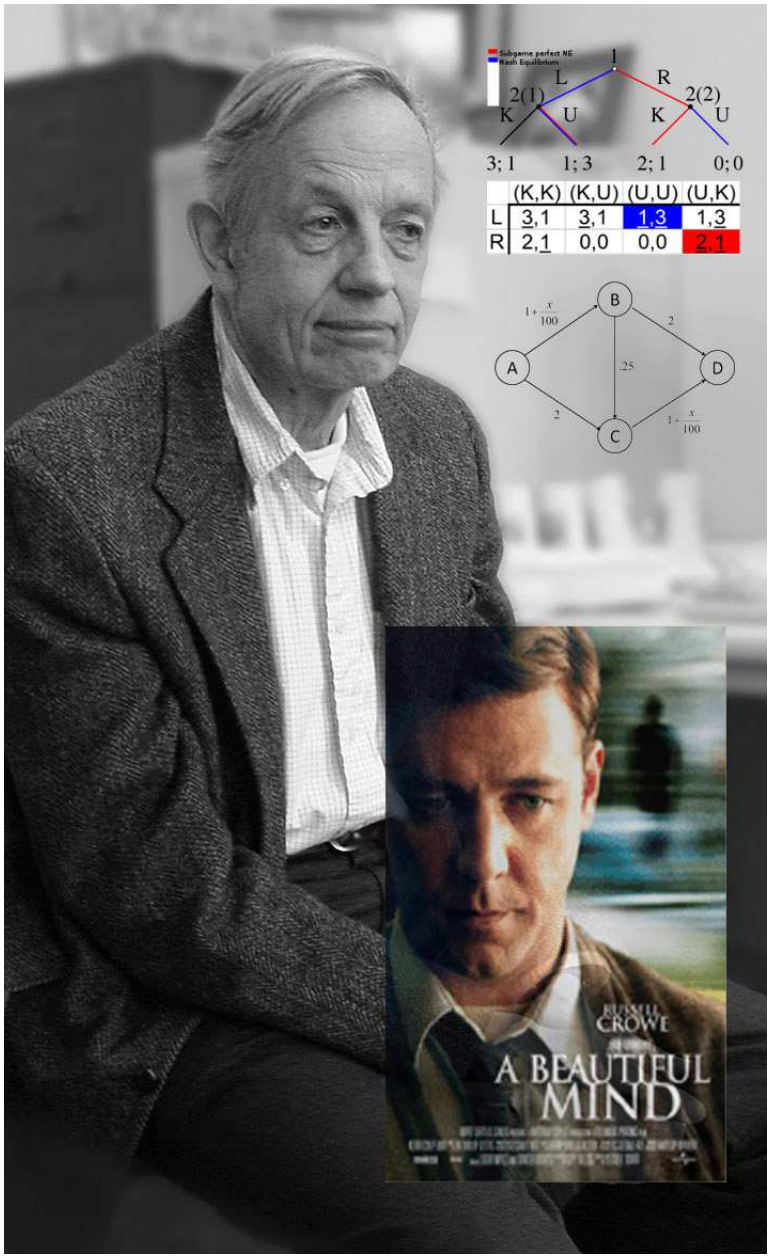


FIGURE 2. Nobel prize winner John Nash, one of the inventors of Hex.

1. Play the game a certain number of times. Record important stages of your games using the boards in Figure 4. Write down your strategies.

2. Try playing against different people.

real-time play, with time settings, and ranking, and http://havannah.vying.org/ for real-time play (Hex/Havannah). Strong game applications include *Six* by Gábor Melis and *Hexy*.

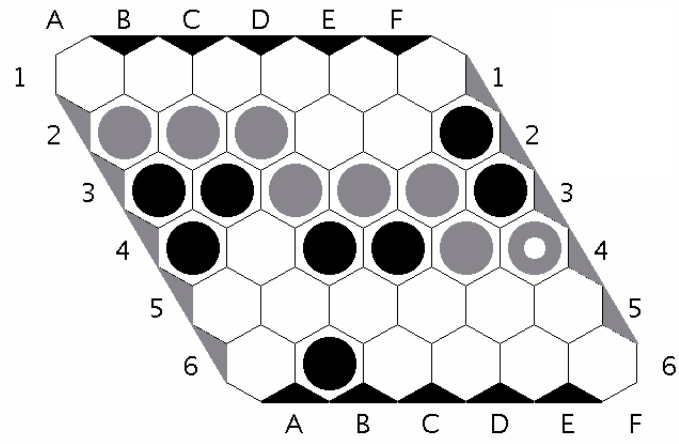


FIGURE 3. Gray has a winning path.

1.2. History of Hex. The game was invented by the Danish mathematician Piet Hein, who introduced the game in 1942 at the Niels Bohr Institute, and also independently invented by the mathematician John Nash in 1947 at Princeton University. It became known in Denmark under the name Polygon (though Hein called it CON-TAC-TIX); Nash's fellow players at first called the game Nash. According to Martin Gardner, some of the Princeton University students also referred to the game as John (according to some sources this was because they played the game using the mosaic of the bathroom floor.). However, according to Sylvia Nasar's biography of John Forbes Nash *A Beautiful Mind*, the game was referred to as "Nash" or "John" after its apparent creator. John Nash was said to have thought of this game, independent of Hein's, during his graduate years at Princeton. In 1952 Parker Brothers marketed a version. They called their version "Hex" and the name stuck. Hex is an abstract strategy game that belongs to the general category of "connection" games. Other connection games include Omni, Y and Havannah. All of these games are related to the ancient Asian game of Go; Nash's version of Hex, in particular, was done as a response to Go.

3. Writing Assignment: Research further information about the people mentioned above or about some of the other "connection" games that are mentioned and write a short paper about your findings.

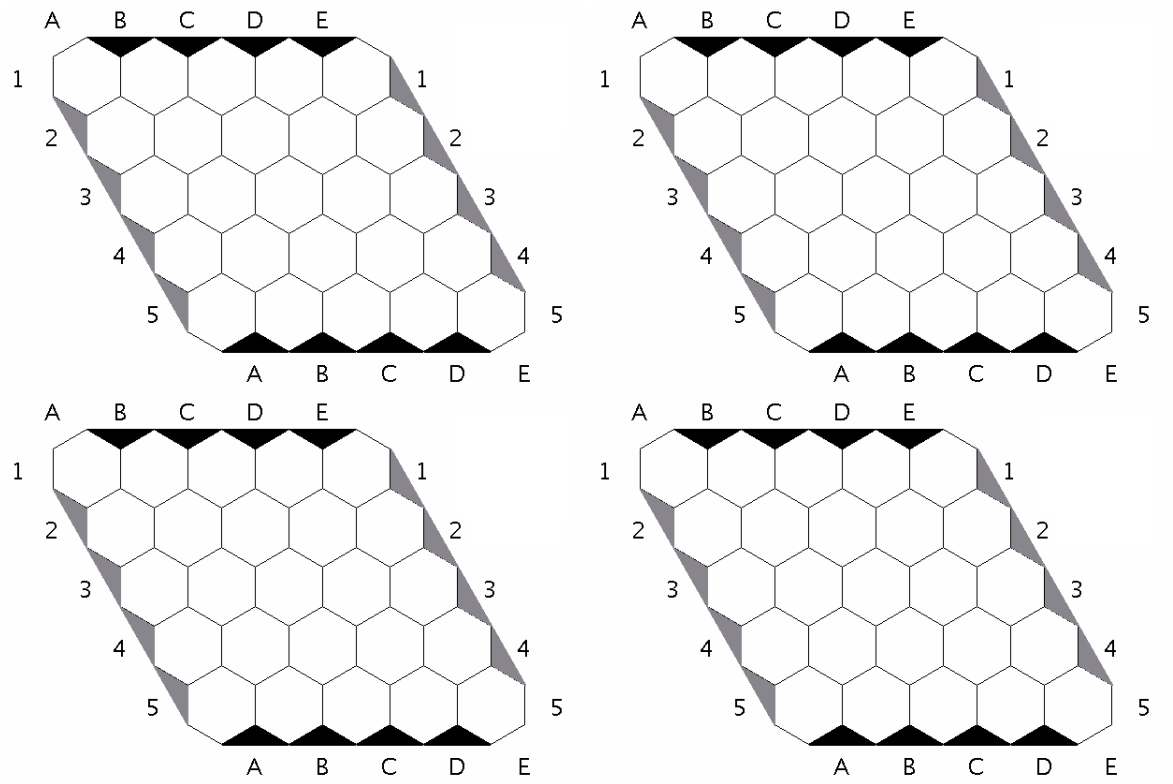


FIGURE 4. Use these 5×5 boards to record important stages of your games.

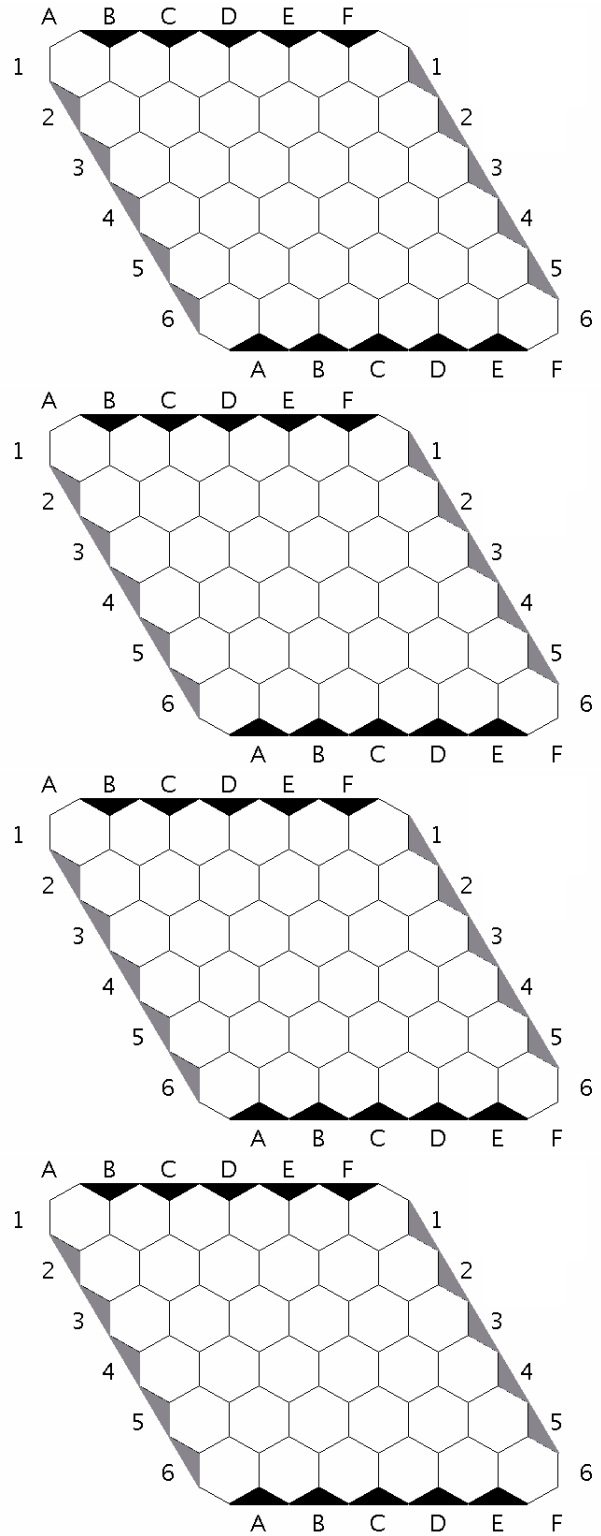


FIGURE 5. Use these 6×6 boards to record important stages of your games.

4. If you have a partner, both of you should speak out loud your reasons for every single move. (While this gives away what you're doing, you will both also learn a lot.)
5. Where do you think is the best place for the first stone? Explain.

1.3. Examples. Consider the board in Figure 6. Notice that the gray stone has a white circle inside it, indicating that this was the most recent move on the board. Black's turn to play.

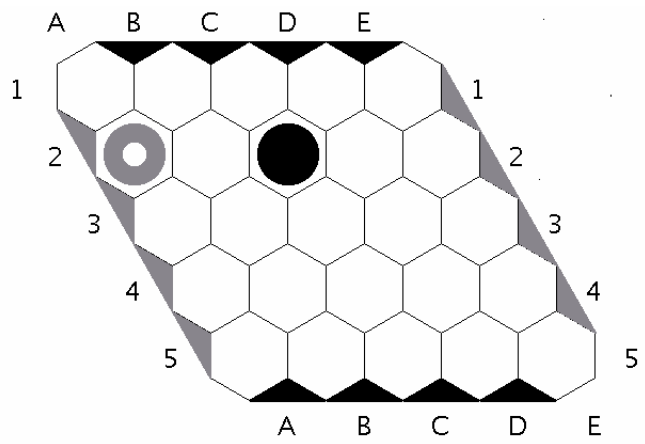


FIGURE 6. Black's turn to play. Which is the best move?

6. Playing for black, where would you place the next stone? Record your moves on the boards of Figure 7.
7. Playing for gray, how would you counter these moves?
8. Continue the game to see whether your choices were good choices.
9. Working with a partner, take turns to explore different strategies.
10. On your own, write down your detailed observations and strategies.

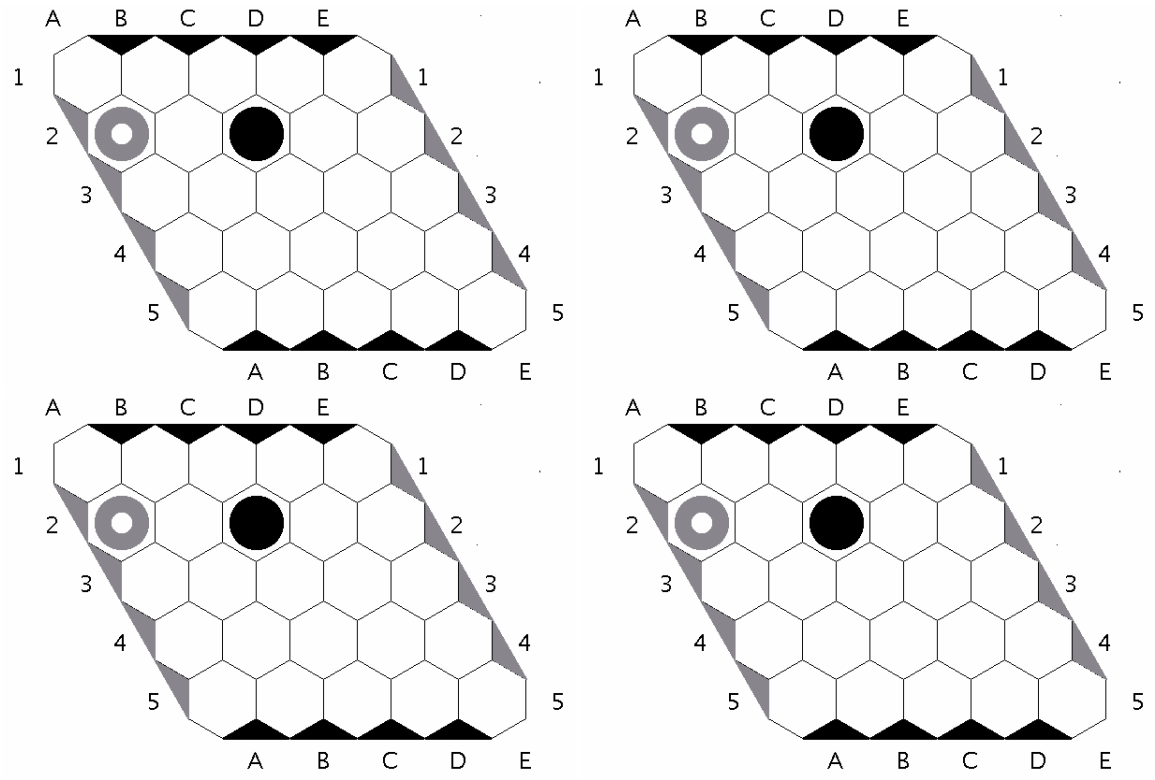


FIGURE 7. Black's turn to play. Which is the best move?

Now consider the board in Figure 8. Black's turn to play.

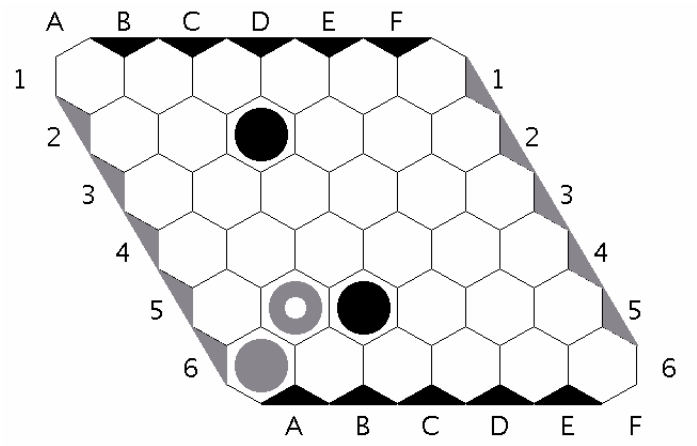


FIGURE 8. Black's turn to play. Which is the best move?

11. Playing for black, where would you place the next stone? Record your moves on the boards of Figure 9.
12. Playing for gray, how would you counter these moves?
13. Continue the game to see whether your choices were good choices.
14. Working with a partner, take turns to explore different strategies.
15. On your own, write down your detailed observations and strategies.
16. **Writing Assignment:** Find a situation in your life outside of school where you needed to use strategies similar to the strategies used in Hex.

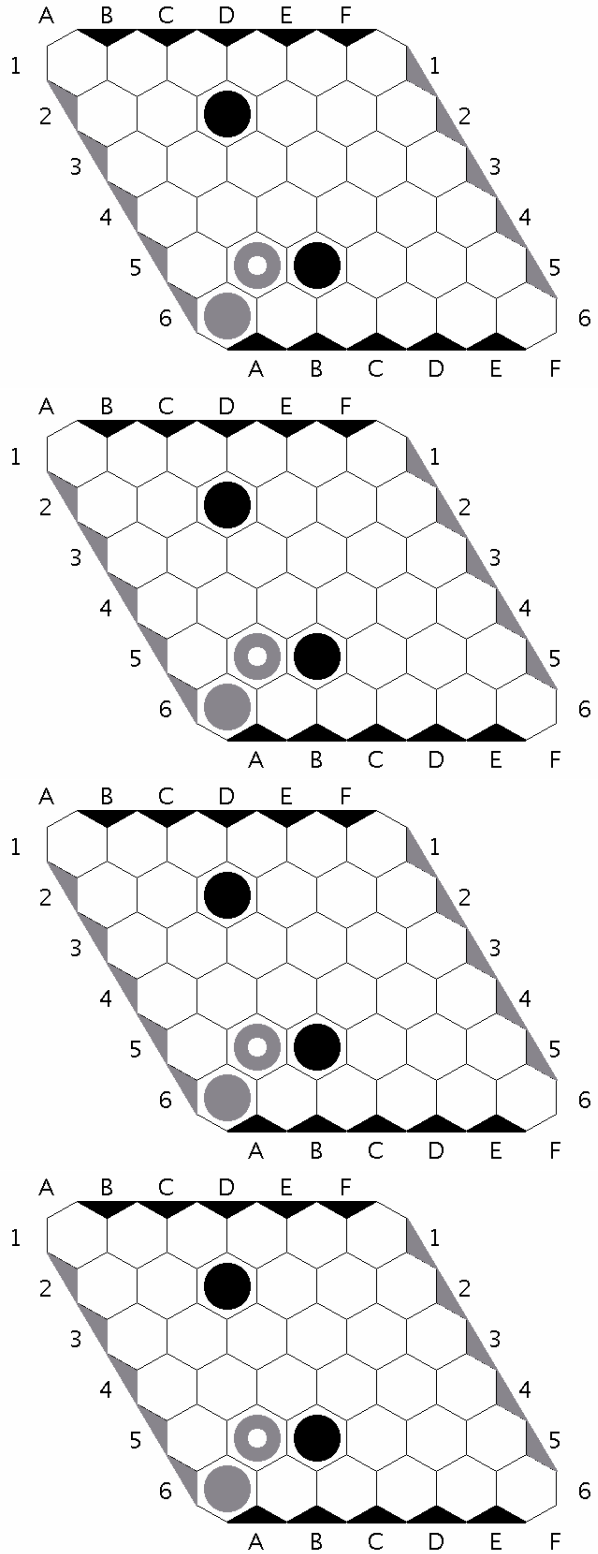


FIGURE 9. Use these boards to record important stages of your games.

1.4. Edge Templates. In order to better understand some strategies underlying Hex, let us consider so-called *Edge Templates*; see Figure 10. **Rules:** Stones can now be placed only in the

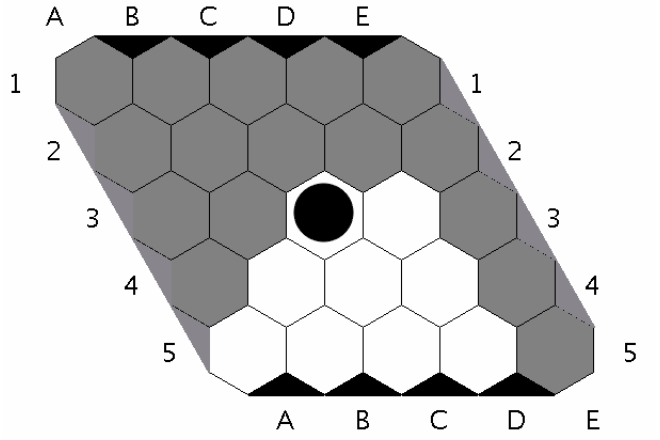


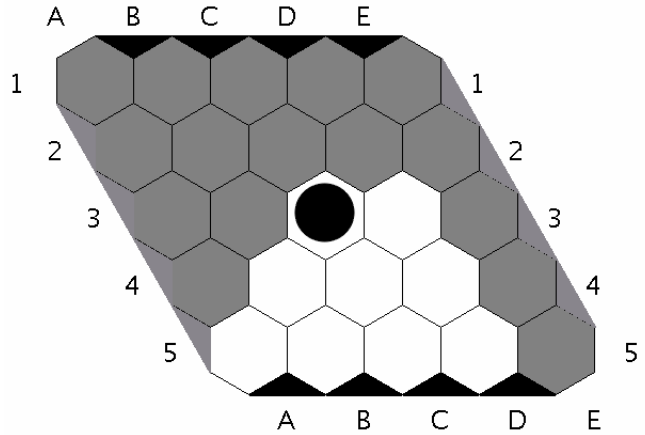
FIGURE 10. How to block Black from reaching the edge?

unshaded hexagons. Black and Gray have different goals: Black's goal is to reach the lower black edge. Gray is trying to block Black from doing so. Gray to go first.

17. With a partner, taking turns, explore the edge template. Use the boards in Figure 11 to record your moves.

18. What do you notice?

19. Is there a pattern in whether Black or Gray is able to win?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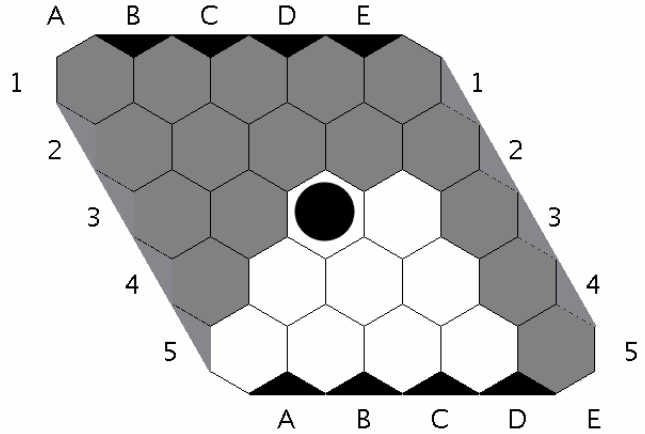
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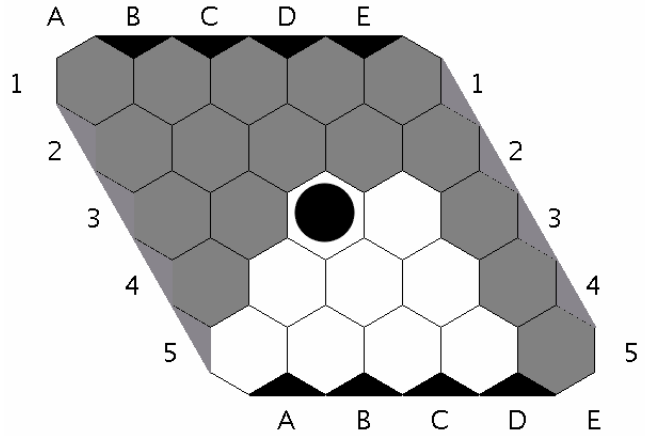
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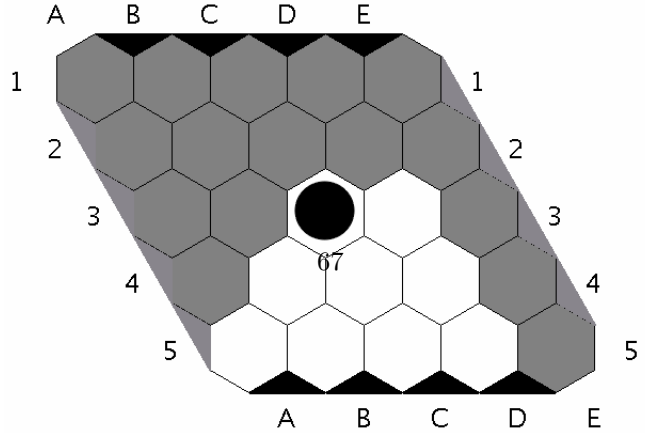
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20. If you notice that one of the players always wins, how can you be sure that this will also happen when playing against other people? Explain.
21. Play against your teacher to check your strategy.
22. How about the board in Figure 12 where Gray has placed a stone to block Black. Playing for Black and using your strategy, how would you respond?

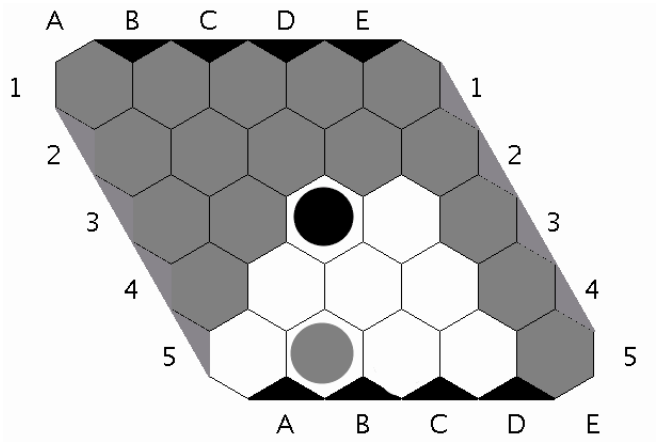


FIGURE 12. Playing for Black, what would your strategy suggest?

Next consider all the above questions for the Edge Template in Figure 13.

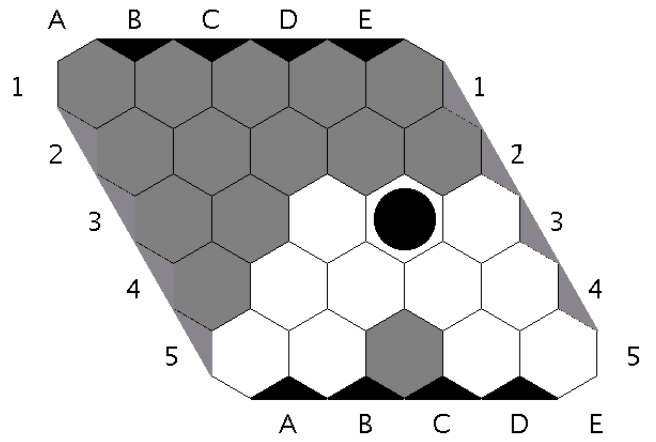


FIGURE 13. How to block Black from reaching the edge?

23. With a partner, taking turns, explore the edge template. Use the boards in Figure 14 to record your moves.
24. What do you notice?
25. Is there a pattern in whether Black or Gray is able to win?

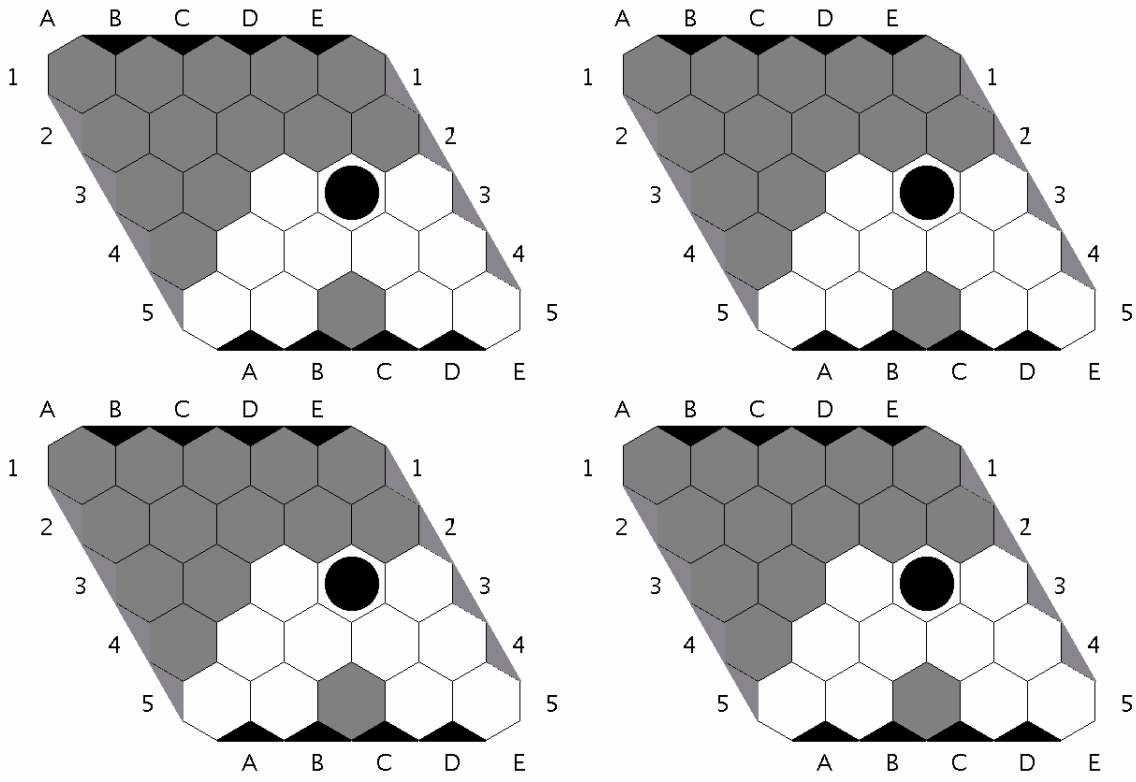


FIGURE 14. Use these boards to record important stages of your games.

26. If you notice that one of the players always wins, how can you be sure that this will also happen when playing against other people? Explain.
27. Play against your teacher to check your strategy.
28. How about the board in Figure 15 where Gray has placed a stone to block Black. Playing for Black and using your strategy, how would you respond?

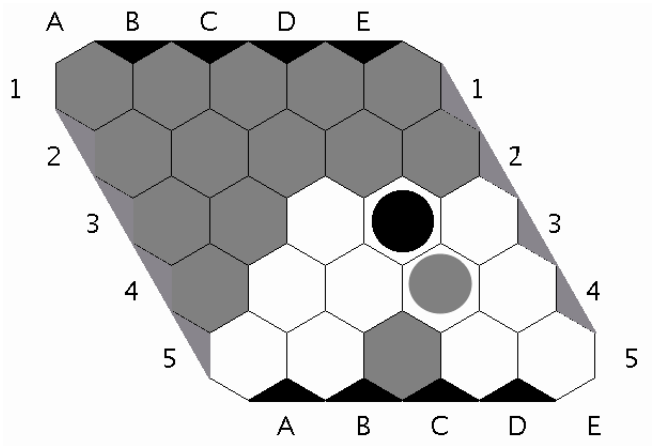


FIGURE 15. Playing for Black, what would your strategy suggest?

Finally, consider all the above questions for the Edge Template in Figure 16.

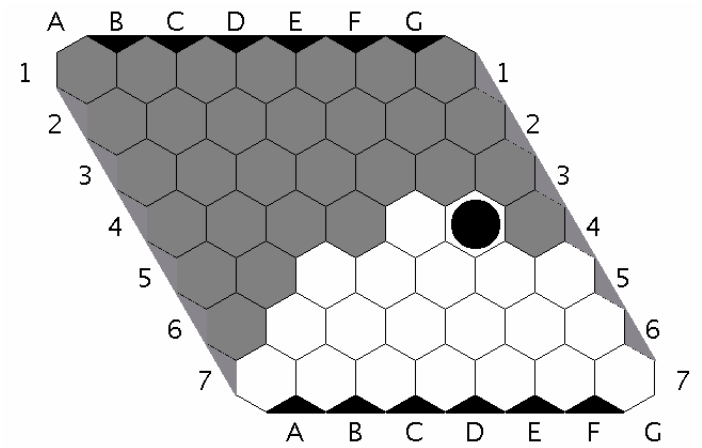


FIGURE 16. How to block Black from reaching the edge?

29. With a partner, taking turns, explore the edge template. Use the boards in Figure 17 to record your moves.
30. What do you notice?
31. Is there a pattern in whether Black or Gray is able to win?

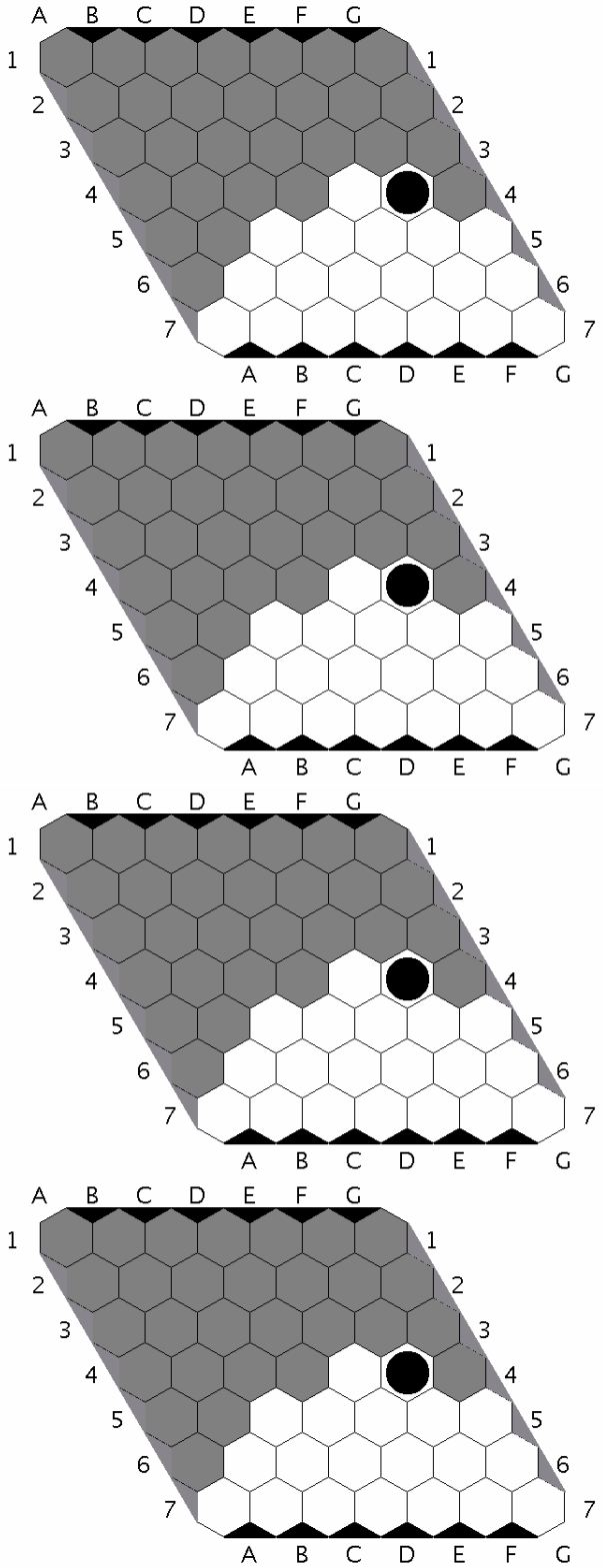


FIGURE 17. Use these boards to record important stages of your games.

32. If you notice that one of the players always wins, how can you be sure that this will also happen when playing against other people? Explain.
33. Play against your teacher to check your strategy.
34. **Challenge question:** How about the board in Figure 18 where Gray has placed a stone to block Black. Playing for Black and using your strategy, how would you respond?

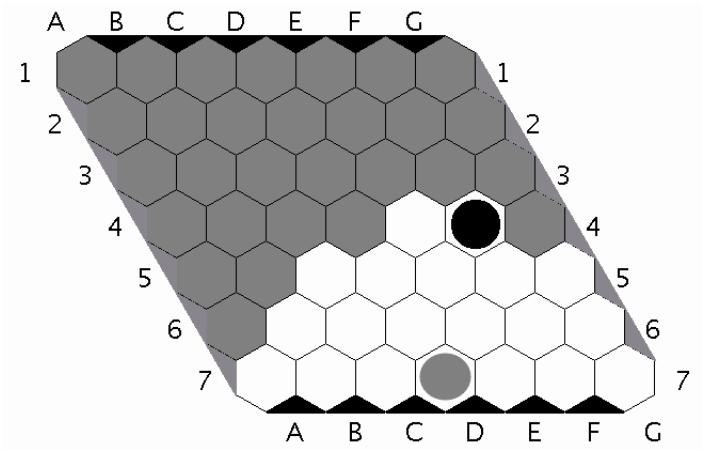


FIGURE 18. Playing for Black, what would your strategy suggest?

1.5. Back to the full board.

35. Having worked with Edge Templates, go back to the full 5×5 or 6×6 Hex boards in Figure 4 and decide where to place the first stone. Explain.
36. Did you use the idea of edge templates in your thinking. Why, or why not.
37. "Skip one:" A student suggests the following strategy: "You always skip one because then you have two options." What could the student mean by that?

1.6. Game Trees – Complexity. We are interested in how many different Hex games we can play on Hex boards of various sizes. As you may have seen, some games end fairly quickly, while others require us to fill up much of the board, so this is a difficult question to get a good handle on. Often in mathematics, when faced with a question that is too difficult, it can be helpful to consider an easier question first.

So let's think about a simpler question first: Imagine we have a new game called "Fill-it-Up" which is just like Hex, except that the players need to continue until the entire board is filled up. Ignore for the moment that the game is not really interesting to play once one person has made a connection.

We consider two "Fill-it-Up" games to be the same when every single move by the two players is exactly the same, in exactly the same order. So two games are different if somewhere along the line, one of the players makes a different move (think of writing down a complete record of every single move: $A1$-$B3$-...; when two of those lists agree, the game is the same, otherwise, it differs).

Now the question I'd like you to consider is: How many different "Fill-it-Up" games are there?

Let's consider for example a 2×2 board. Black goes first and puts a stone in $A1$. Next, gray places a stone in $A2$. Followed by black in $B2$ and gray in $B1$. Now the board is completely filled so we stop. A different example: $B2$-$A2$-$A1$-$B1$ (black starts and player take turns). Notice that once all the moves are done, the boards look identical, but we got there in different ways, so

38. In how many ways can you fill up a 2×2 board (taking turns between Gray and Black)?
39. How many possibilities do you have to place the first stone?
40. Given that there is now one stone on the board, how many possibilities do you have to place a second stone?
41. Given that there are two stones on the board, how many possibilities do you have to place the third stone?
42. What pattern do you observe? Explain.
43. Now consider a 3×3 board? What would the pattern look like here? In how many ways can you fill up this board?
44. How about a 4×4 board?
45. Describe a process to find out in how many ways we could fill up a 14×14 board.
46. How does the number of different "Fill-it-Up" games relate to the total number of possible **Hex** games? (Which is what we are really interested in.)

Size of board n	Number of hexagons on $n \times n$ board	Number of ways to fill up the board
$n = 2$		
$n = 3$		
$n = 4$		
$n = 14$		

We may need a calculator, or even a computer with Maple or Mathematica to work out these numbers in detail.

This method over-counts the total number of possibilities. Cameron Browne shows a more careful accounting, with the following numbers.

Size of board n	Valid Game Positions
$n = 1$	2
$n = 2$	17
$n = 3$	2844
$n = 4$	4,835,833

47. How could you program a computer to win in every game?

1.7. Cool connections of Hex with the non-mathematical world.

- iPhone App: Hexatious

2. Appendix: Game Boards

3. ConHex

History: The game itself was created in 2002 by Michael Antonow, who really brought together a few different concepts in this two-player, abstract strategy game. From: Calvin Daniels, http://www.boardgamegeek.com/thread/434435/

Other connection games: Stymie, Y, Twixt, (see Cameron's Browne's other book)

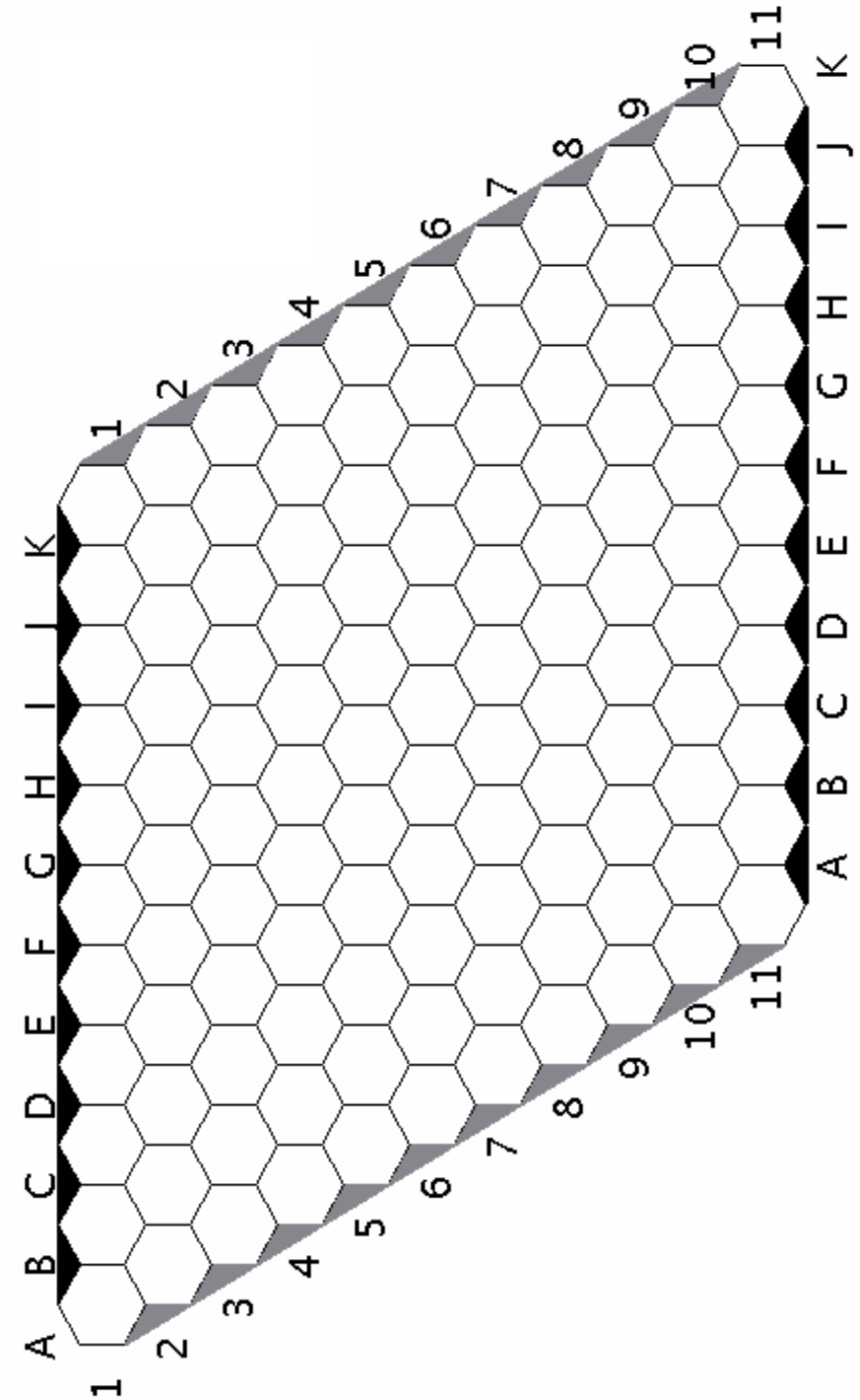


FIGURE 19. A regular 11×11 Hex board.

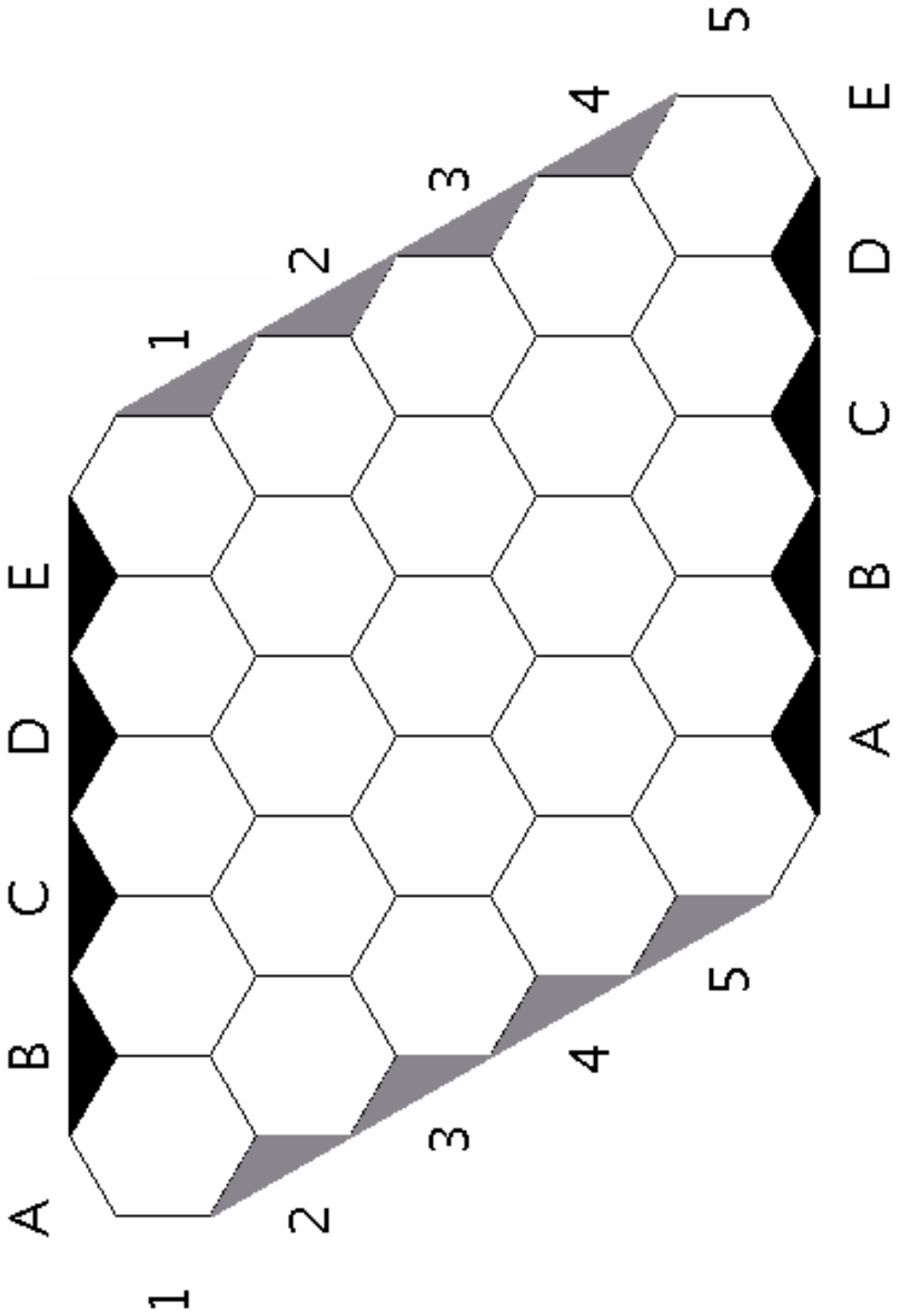


FIGURE 20. A regular 5×5 Hex board.

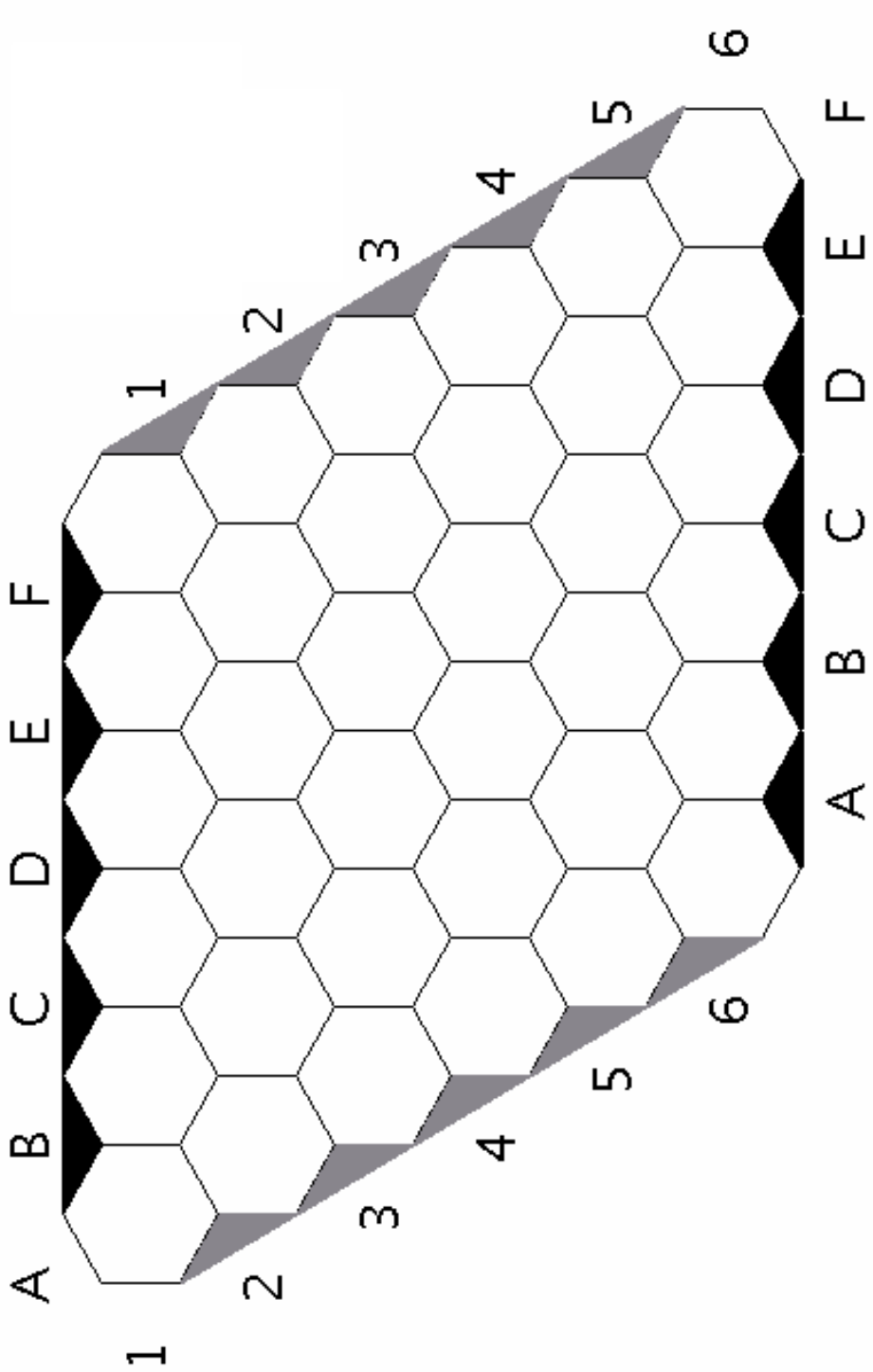


FIGURE 21. A regular 6×6 Hex board.

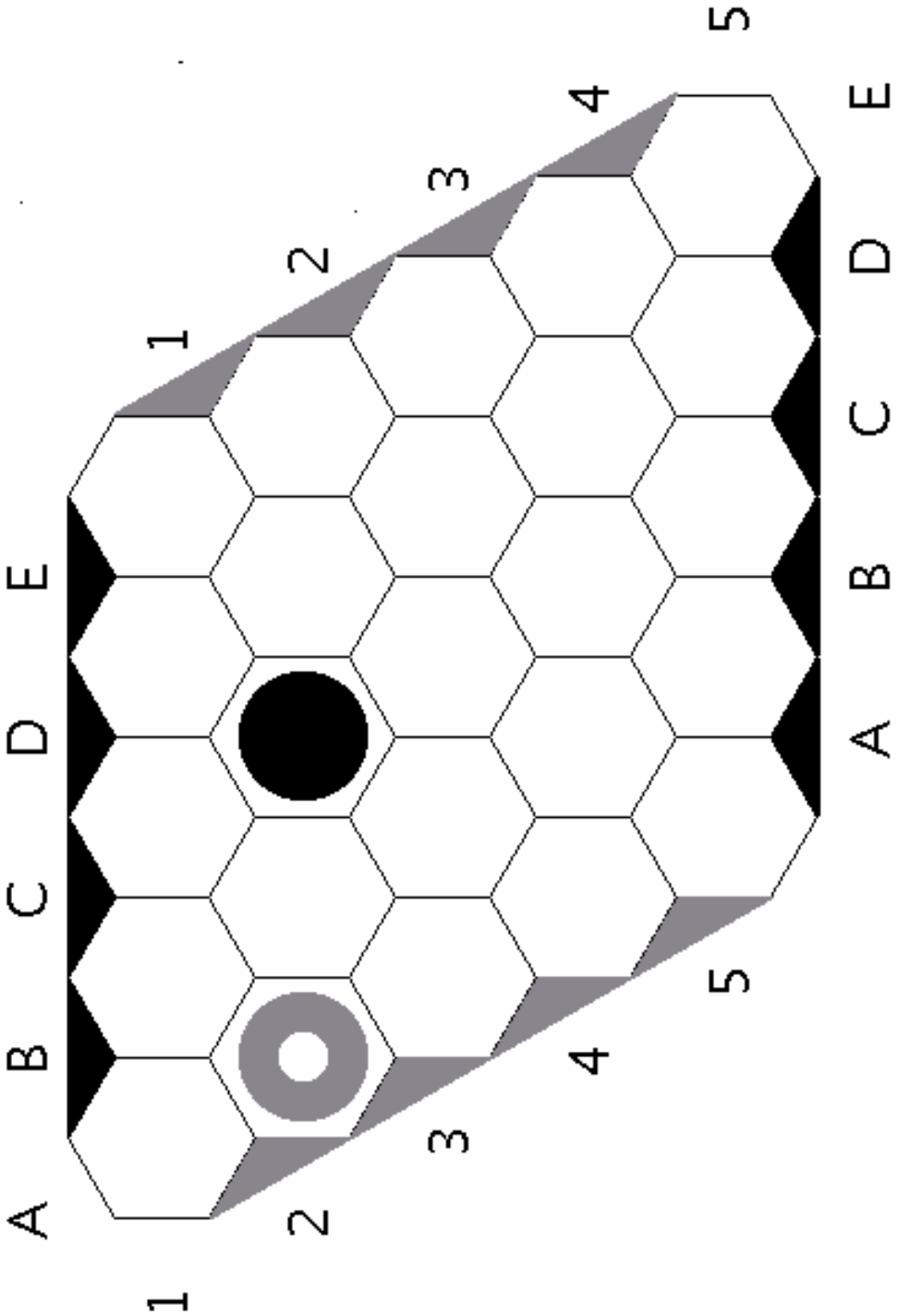


FIGURE 22. Black's turn to play. Which is the best move?

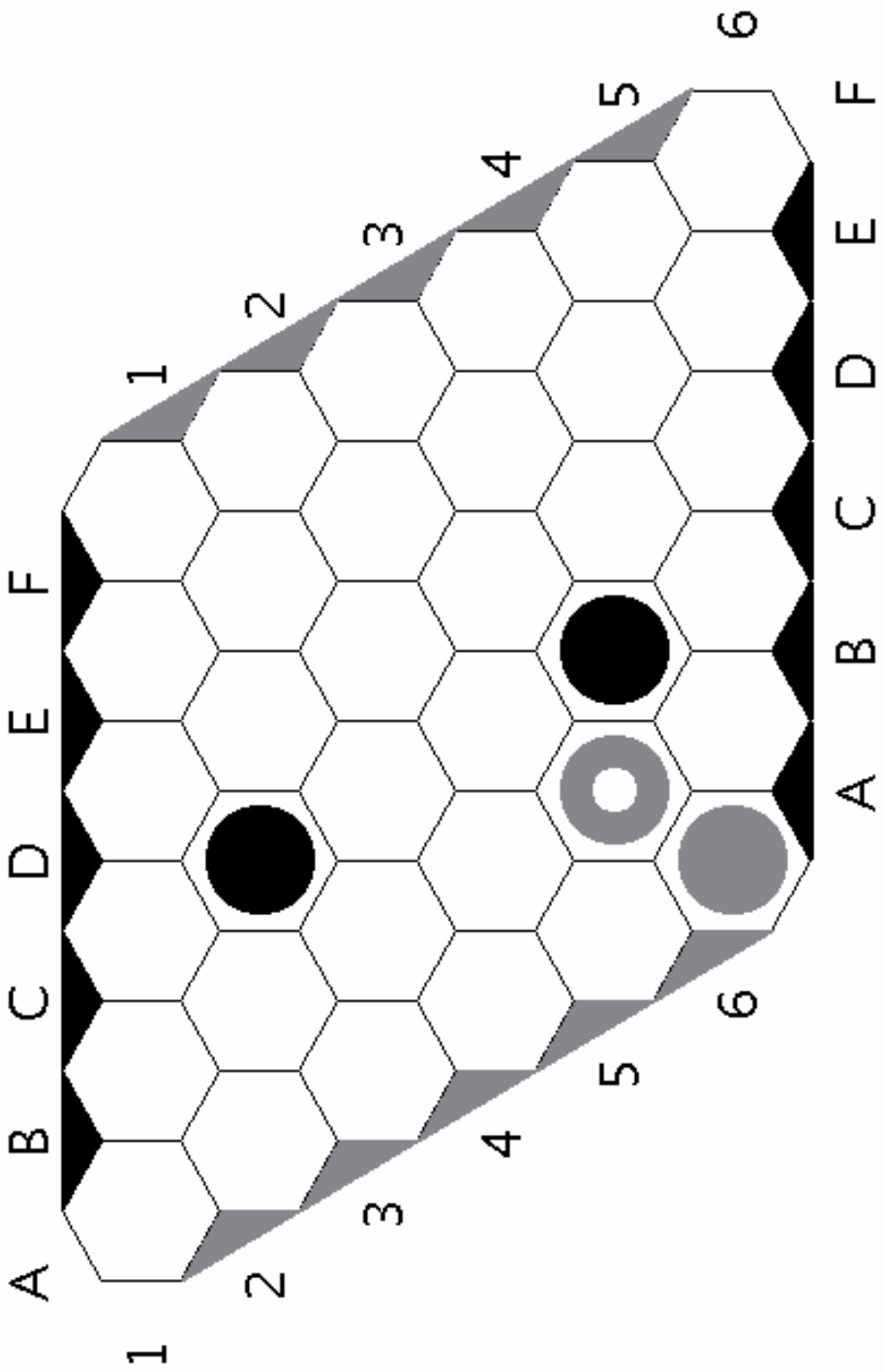


FIGURE 23. Black's turn to play. Which is the best move?

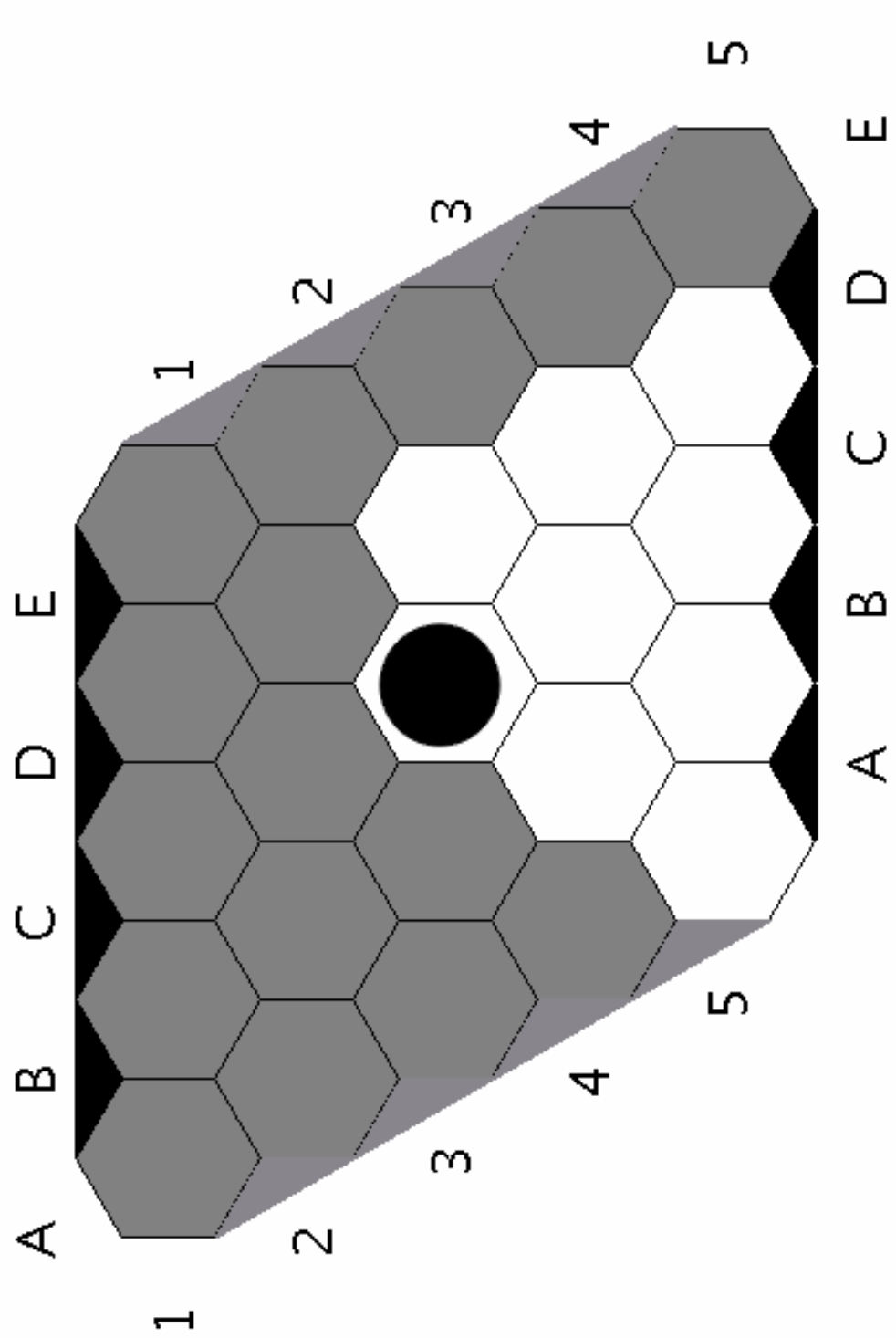


FIGURE 24. How can Gray block Black from reaching the edge?

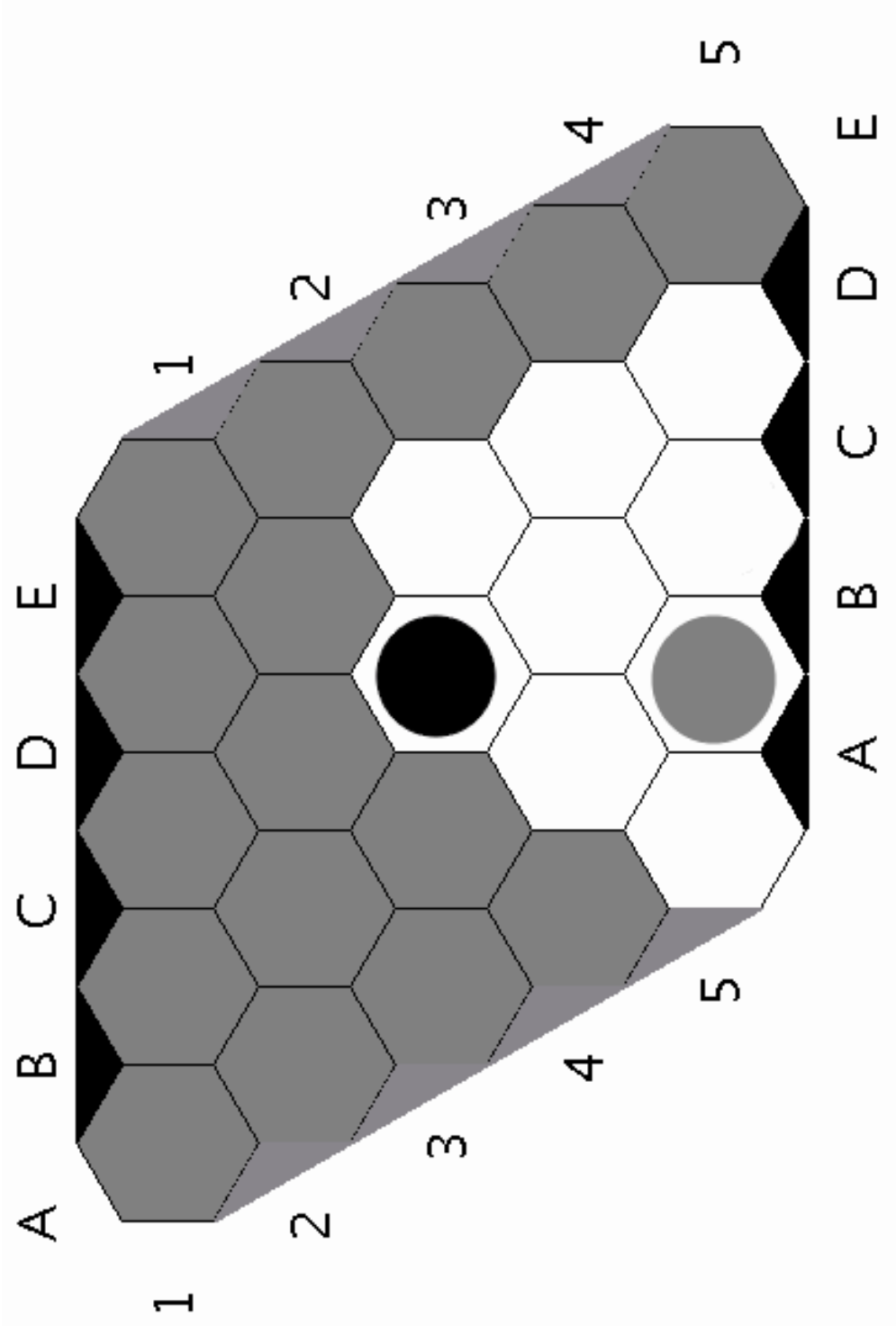


FIGURE 25. Playing for Black, what would your strategy suggest?

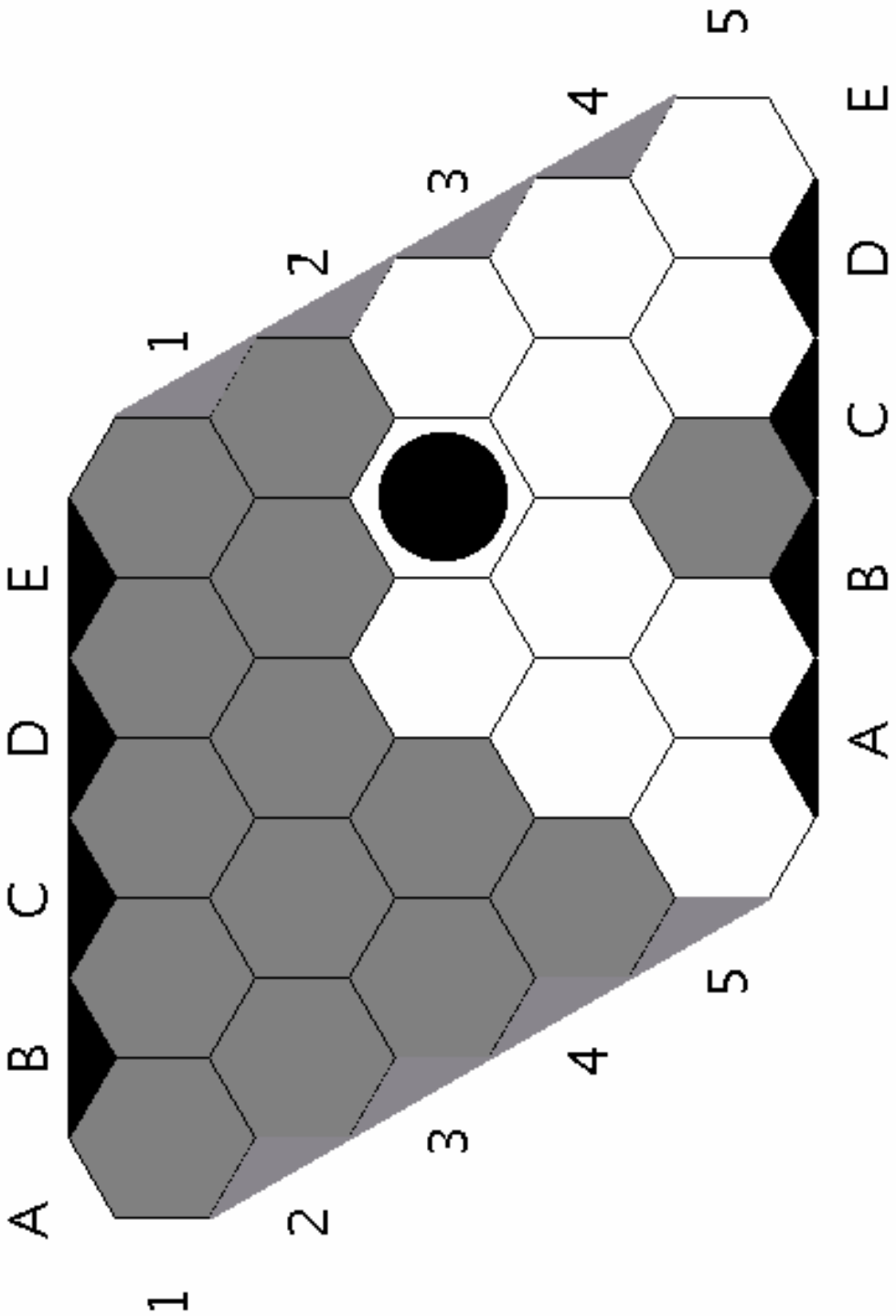


FIGURE 26. How can Gray block Black from reaching the edge?

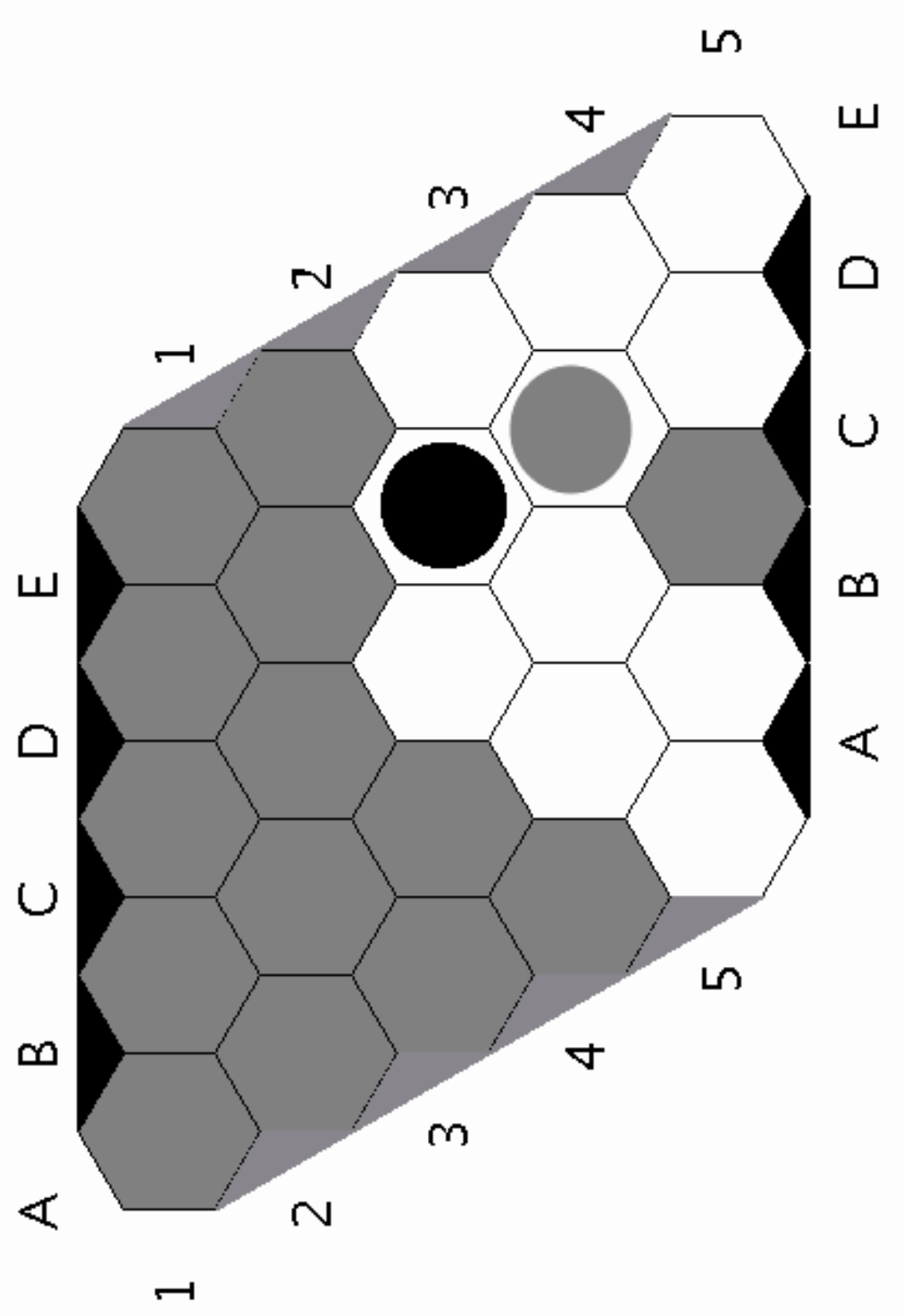


FIGURE 27. Playing for Black, what would your strategy suggest?

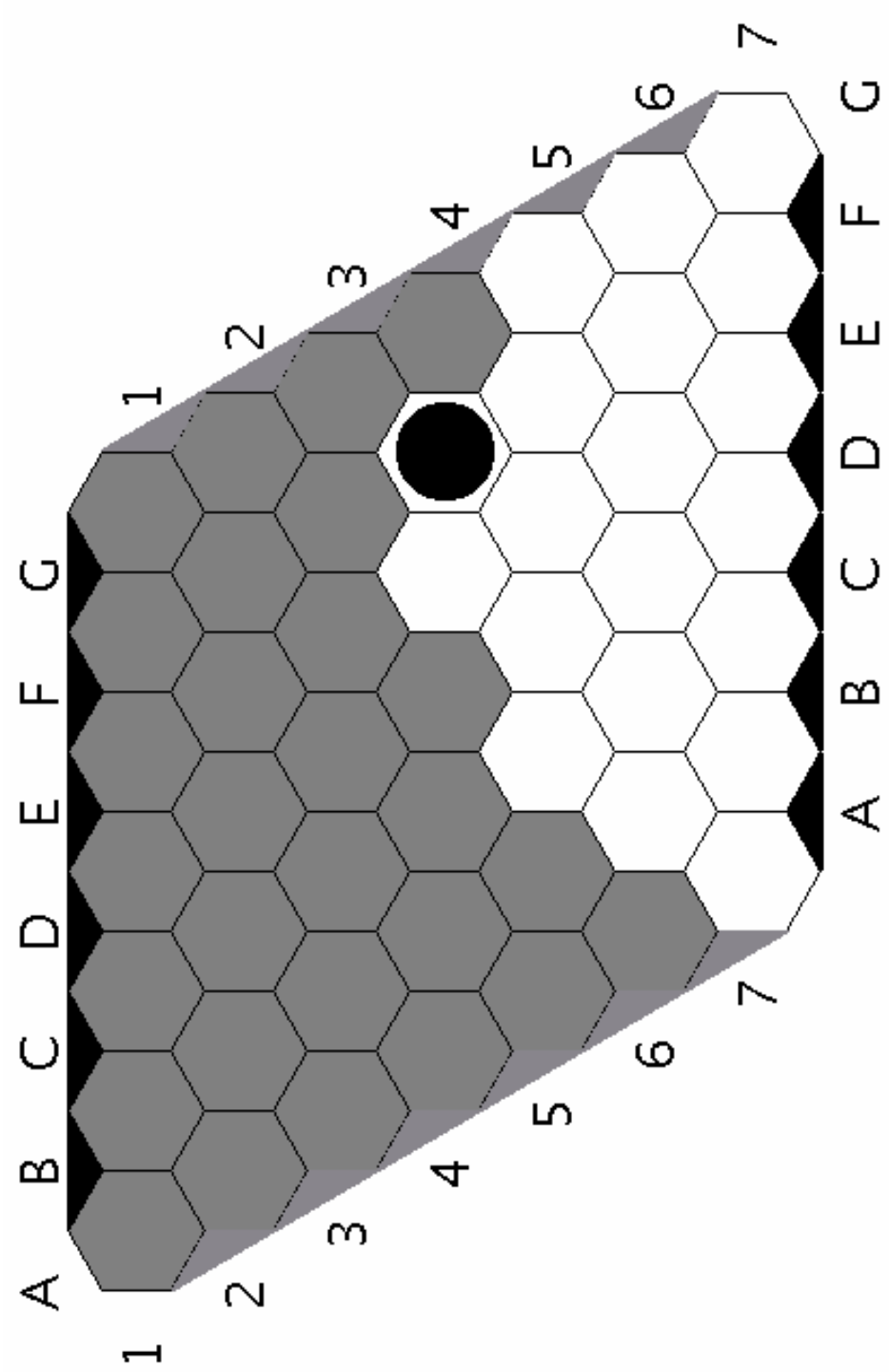


FIGURE 28. How can Gray block Black from reaching the edge?

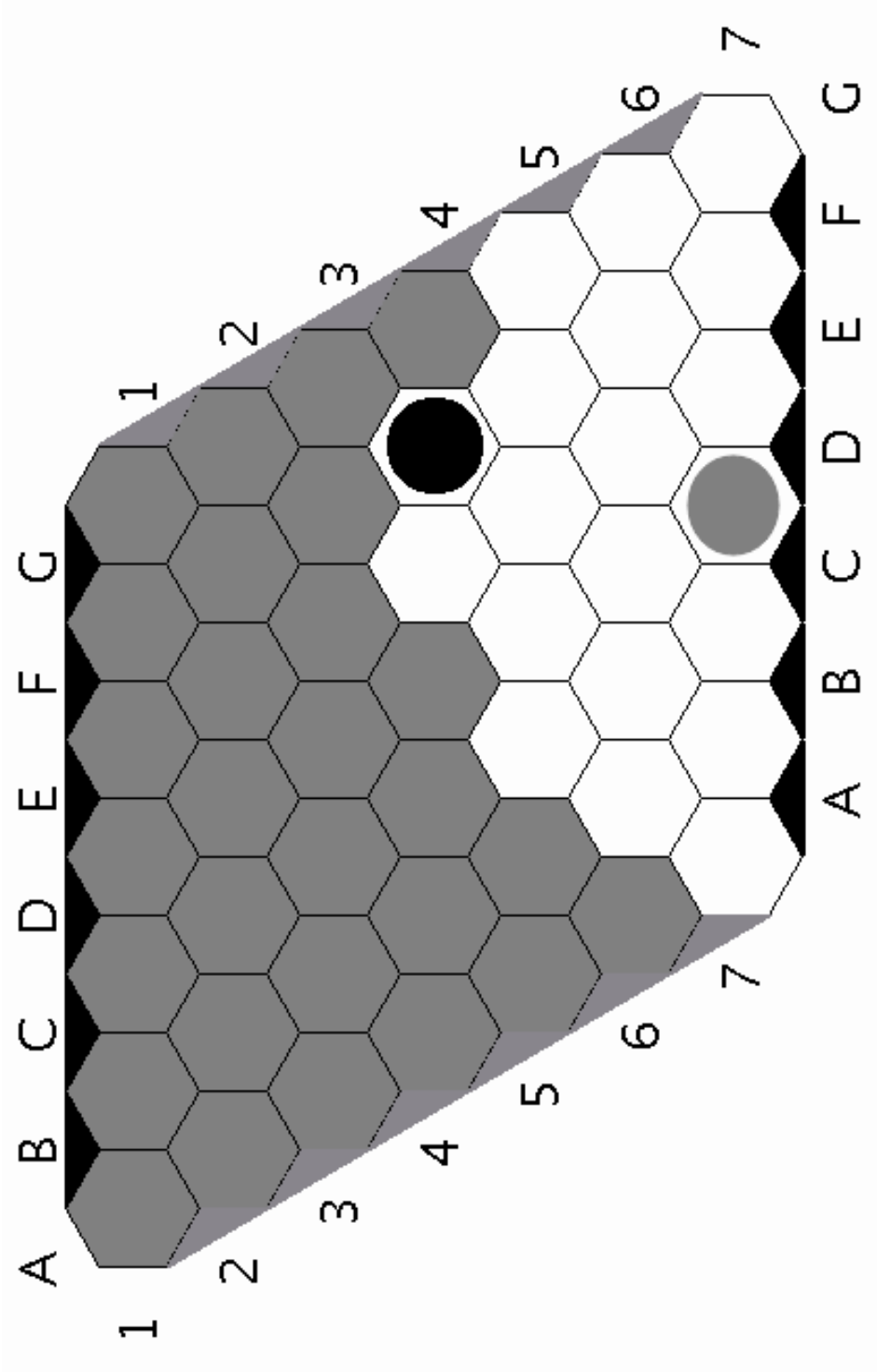


FIGURE 29. Playing for Black, what would your strategy suggest?

CHAPTER 5

One pile, two pile, three piles

1. One pile

Rules: "One pile" is a two-player game. Place a small handful of stones in the middle. At every turn, a player decides whether to take one, two, or three stones from the pile. Whoever takes the last stone(s) wins.

1. Play the game a few times with a partner.
2. At what point of the game do you know if you've lost or won? Explain why. How many stones were left in the pile when you knew that?
3. Can you find a higher number of stones so that you're sure you will win or lose?
4. How about the next highest number of stones? Is there a pattern? (How high can you go?)
5. Suppose there are 4 stones in the pile. Do you want to go first or second? Explain why.
6. Suppose there are 9 stones in the pile. Do you want to go first or second? Explain why.
7. Suppose there are 15 stones in the pile. Do you want to go first or second? Explain why.
8. Explain the full strategy of the game (a "strategy" tells you precisely what to do in any situation).
9. Now that you know the strategy, is this game still interesting to play?
10. Now we change the rules of the game so you can remove up to five stones at each turn. What is your strategy now?
11. Now the players are allowed to remove up to thirteen stones at each turn? What is your strategy now?

2. Two piles

Rules: Now, we place *two* piles of stones in the middle. Players take turns. At each turn, you get to pick a pile and take as many stones as you want from that pile. Whoever takes the very last stone(s) wins.

Example: For example, I could take one whole pile. But then my opponent could take the other pile and I would lose.

12. Play the game a few times with a partner.
13. At what point of the game do you know if you've lost or won? Explain why.
14. Can you find a situation (i.e. two piles with a certain number of stones each) so that you're sure that you will lose? Explain why.
15. Can you find a situation with higher numbers of stones so that you're sure that you will lose? Explain why.
16. Suppose there are 2 stones in each pile. Do you want to go first or second? Explain why.
17. If you feel that you have a good strategy, try the following two situations. If not, take a look at Investigation **20** and come back later.
  a. Suppose there are 7 and 9 stones in the respective piles. Do you want to go first or second? Explain why.
  b. Suppose there are 6 and 25 stones in the respective piles. Do you want to go first or second? Explain why.

18. Explain the full strategy of the game (a strategy tells you precise what to do in any situation).
19. Now that you know the strategy, is this game still interesting to play?
20. (Tree) Consider the situation in Figure 1 with 2 stones in one pile, and one stone in the other. Anna plays against Ben. Anna gets to go first. The following tree lists all of Anna's possibilities for her first move, and one scenario where Ben ends up winning. Fill in the missing scenarios.

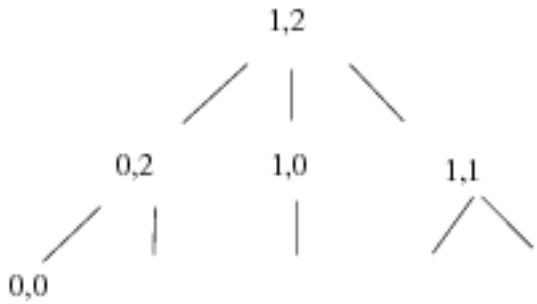


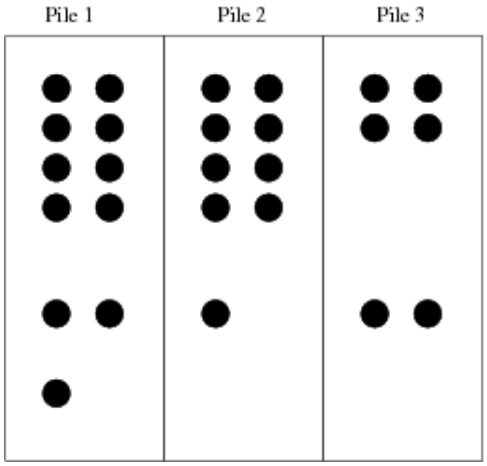
FIGURE 1. Anna

21. Which move should Anna start with in order to win? Explain.
22. Use a similar tree to analyze the game starting with two stones in both piles.
23. Are there any subtrees that you have already seen before? Which ones?
24. **Classroom Discussion:** What precisely do we mean when we say "Anna has a winning strategy?" How can you use winning strategies to create shortcuts in your tree notation?
25. Next, analyze the game tree for a game with two stones in one pile and three in the other, i.e. starting with $2, 3$. Use shortcuts whenever possible.
26. Analyze the $3, 3$ game using a tree.
27. **Classroom Discussion:** Bringing together our understanding from Investigations **17a-17b** and our game tree investigations, what is a winning strategy for "Two Piles"?

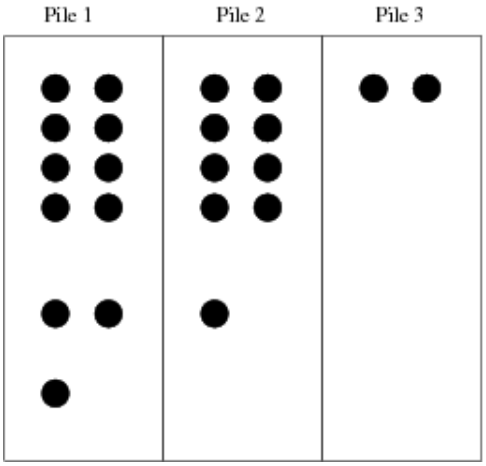
3. Three piles

Rules: Now, we place *three* piles of stones in the middle. Players take turns. At each turn, you get to pick a pile and takes as many stones as you want from that pile. Whoever takes the last stone(s) wins.

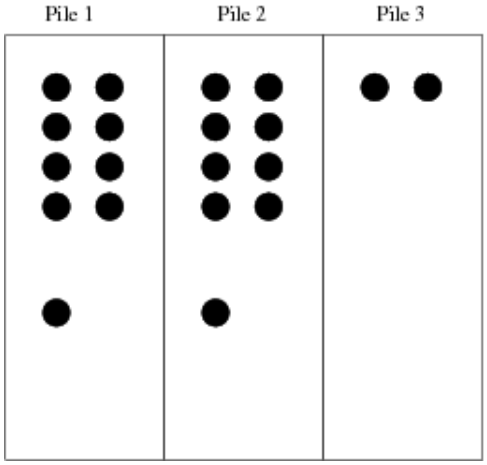
28. Play the game a few times with a partner. Start with small numbers of stones.
29. At what point of the game do you know if you've lost or won? Explain why.
30. Can you find a situation so that you're sure that you will lose? Explain why.
31. Can you find a situation with higher numbers of stones so that you're sure that you will lose? Explain why.
32. Suppose there are 1, 2, and 2 stones in each pile. Do you want to go first or second? Explain why.
33. Suppose there are 1, 2, 3 stones respectively in each pile. Do you want to go first or second? Draw a game tree to explain your strategy.
34. Suppose there are 1, 2, 4 stones respectively in each pile. Do you want to go first or second? Explain why. Draw a game tree to explain your strategy.



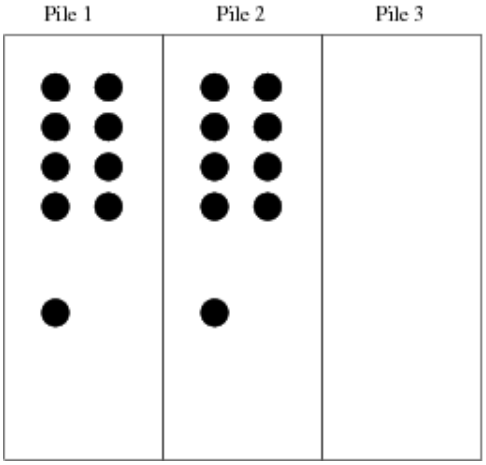
(a) Amanda arranges the three piles.



(b) Amanda takes 4 stones from pile three.



(c) Jim decides to take 2 stones from pile one.



(d) Amanda decides to take two stones from pile three.

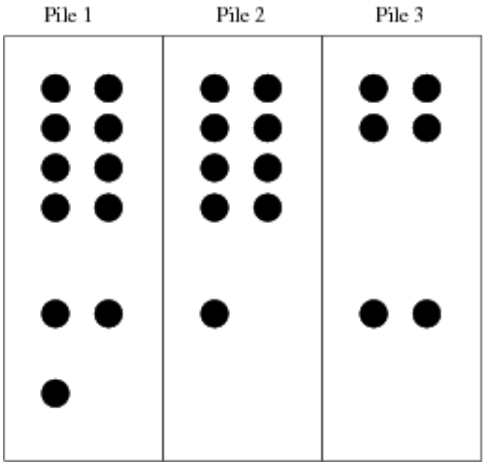
FIGURE 2. First game starting with 11-9-6.

Watching a game: Figures 2(a)-2(d) show details of a Three Piles game between Jim and Amanda. The piles hold 11, 9, and 6 stones, respectively. It's Amanda's turn. Notice how she arranges the stones in Figure 2(a).

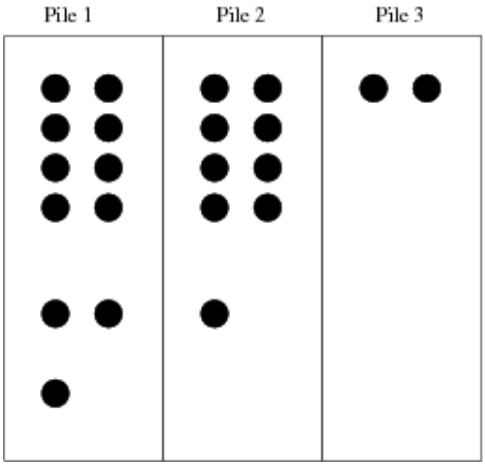
35. What is Amanda doing? Describe any patterns you notice in the arrangements.
36. Who is going to win the game? Explain why.
37. When asked, Amanda reports: "I pair up subpiles." Explain what she might mean by that.

In Figures 3(a)-3(d), we see another game. Amanda starts out in the same way, but now Jim is making different choices.

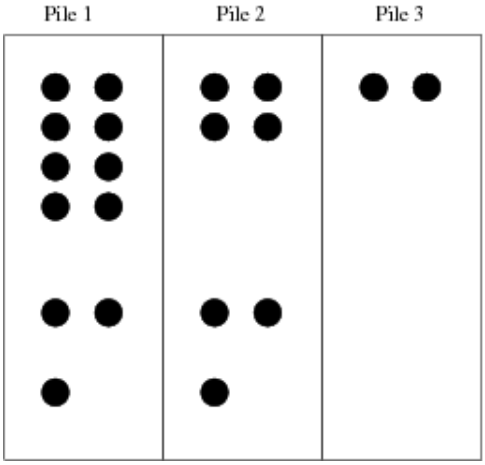
38. What is Amanda doing? Describe any patterns you notice in the arrangements.
39. Who is going to win the game? Explain why.



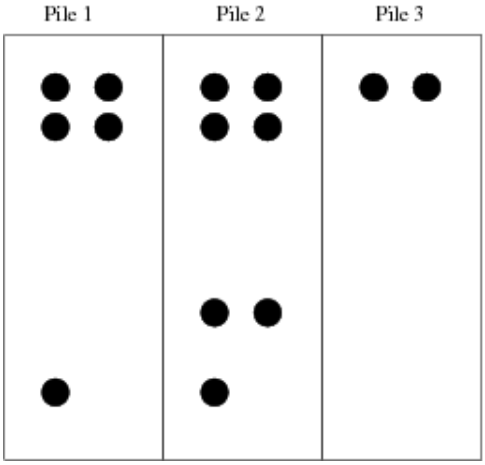
(a) Amanda arranges the three piles.



(b) Amanda takes 4 stones from pile three.



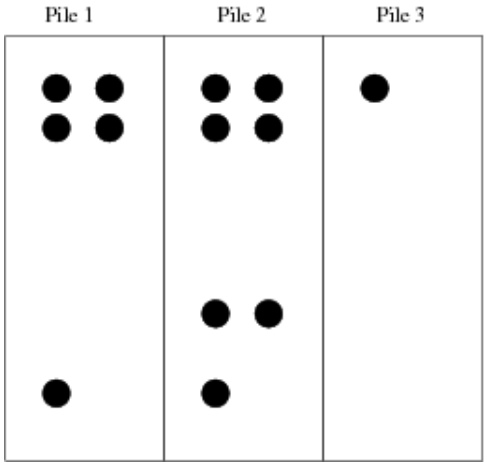
(c) Jim decides to take 2 stones from pile two. Amanda rearranges the piles as shown.



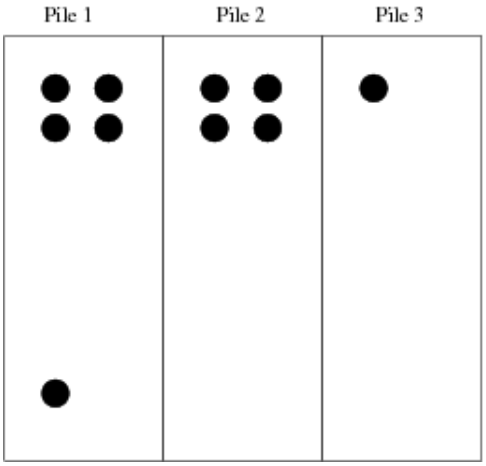
(d) Amanda decides to take 6 stones from pile one.

FIGURE 3. Second game of Three Pile between Jim and Amanda, first steps.

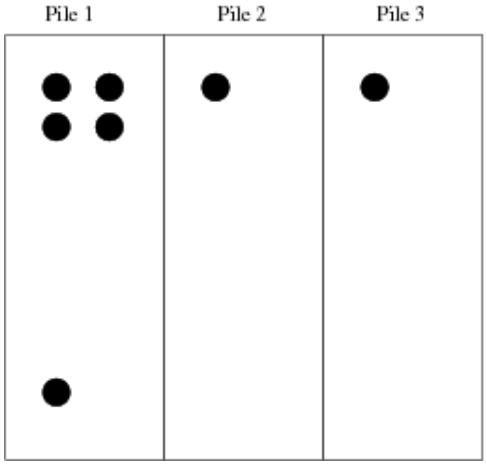
40. What can you say about Amanda's strategy of "pairing up subpiles?" Explain what she might mean by that.



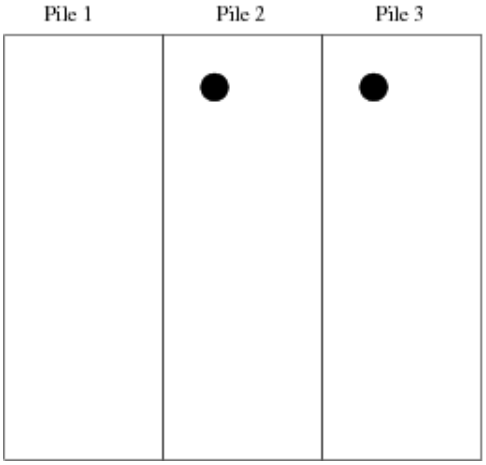
(a) Jim decides to take 1 stone from pile three.



(b) Amanda decides to take 3 stones from pile two.



(c) Jim decides to take 3 stones from pile two.



(d) Amanda decides to take all stones from pile one.

FIGURE 4. Second game of Three Pile between Jim and Amanda, continued.

41. With your partner, explore some of the strategies you observed. Play a few games, keep track in your notebook of your moves, stone arrangements, and strategies.

42. Once you feel clear about your strategies, determine your first move for the following arrangements. Explain your strategy.

a. Three piles with 10, 7, 5 stones, resp.

b. Three piles with 8, 7, 4 stones, resp.

41. So far we have seen subpiles of size 1, 2, 4, and 8. How do you think this sequence will continue? Explain.

42. Carlos does not want to draw pictures into his notebook for a game with 7, 5, 2 stones. Instead, he writes $7 = 2 + 2 + 2 + 1$, $5 = 4 + 1$, $2 = 2$ to show his subpile arrangement. It is his turn. He is convinced that he will win by taking 4 from the second pile. What do you think is his reasoning?

43. Do you think he is correct? Explain.

44. His opponent, Amanda writes the remaining subpiles as $7 = 4 + 2 + 1$, $1 = 1$, $2 = 2$. She is convinced that *she* will win, by taking 4 from the first pile. Do you think she is correct? Explain.

45. What important aspect of the winning strategy does this example reveal?

46. Consider the following game position, with piles of size 15, 25, and 12. Use the table to determine the subpiles, and decide whether you want to go first or second with this game.

Pile size	16	8	4	2	1
15	0	1	1	1	1
25					
12					

47. Now consider the following game position, with piles of size 42, 17, and 57. Use the table to determine the subpiles, and decide whether you want to go first or second with this game.

Pile size	16	8	4	2	1
42					
17					
57					

48. Writing Assignment: The family of games we have described so far is actually called *Nim*. Do some research into this game, its history, and its strategies. Write about your findings.

3.1. Why does it work? This brief section summarizes the main ingredients that can allow us to prove why the "subpile" strategy will always work.

- Every number can be expressed in a unique way as a binary number.
- Express piles sizes as binary numbers.
- Add each of the slots using *Nim-sum*: $1 + 1 = 0$!

49. For each of the games you have worked on so far, determine the Nim-sum at each turn. What kind of patterns do you observe?

50. Classroom Discussion: What is the connection between the subpile strategy and the Nim-sum?

F1. Northcott's Game: On every row of a rectangular board, there are two checkers: one white and one black. A move consists sliding a single checker in its original row without jumping over another checker. The player to make the last move wins. See Figure 5 for an example game board. Use your knowledge of Nim to find a winning strategy for Northcott's game. [1]

[1] For your exploration, you can find online applets of Northcott's game where you can play against a computer.

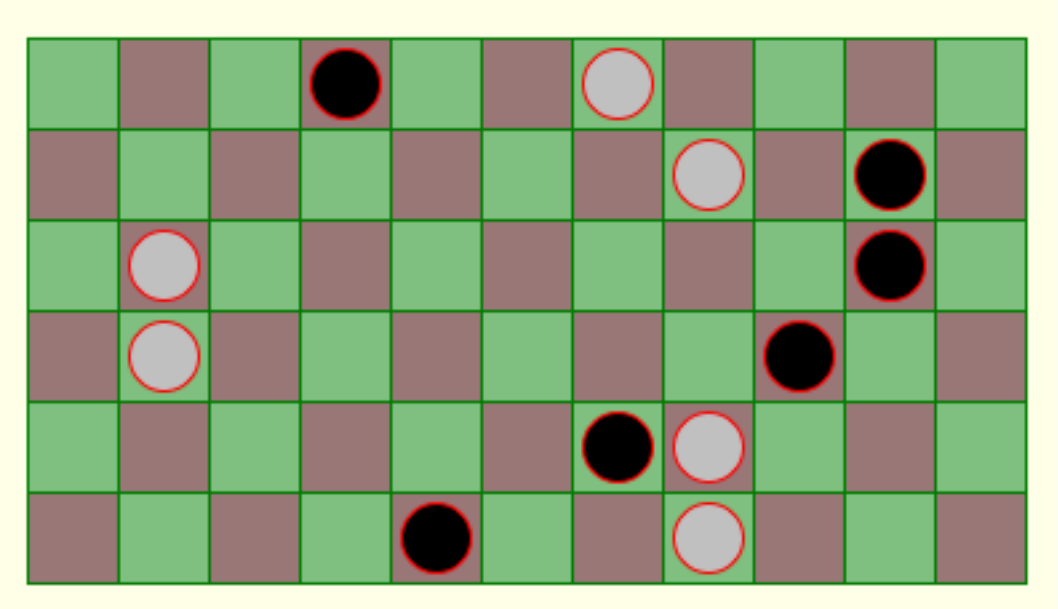


FIGURE 5. Northcott's game position.

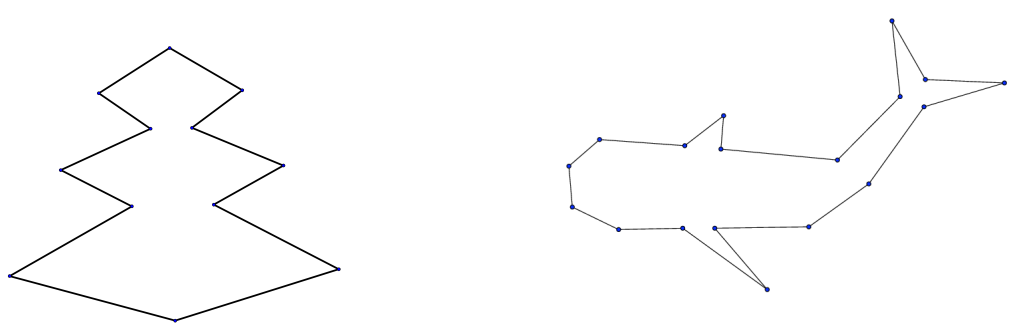
CHAPTER 6

Straight-cut Origami

1. The Fold-and-cut Problem

1. Take a piece of paper, fold it up in any way you like, as long as the end result is flat. Now make one complete straight cut (i.e., a cut along a line). Unfold the pieces, and see what you get. Try a few examples. Share with your class.
2. Describe a few characteristics that all these shapes have in common.
3. Will any and all shapes obtained by this process have these characteristics? Explain your reasoning in detail.
4. In general, what types of shapes do you think can be obtained in this way? Be as precise as you can.
5. What kinds of shapes are *not* possible? Describe a few examples. Explain why you are convinced that they are not possible.

In this chapter, we will investigate the **Fold-and-cut Problem**, sometimes also called the problem of Straight-Cut Origami: Given a shape, such as those in Figure 1, is there a a way to fold it up so that you can cut it out with a single straight cut? If there is such a way, how do we actually fold it? How can we describe all the shapes for which this is possible?



(a) Japanese *sangabisi* crest (large: Figure 13). (b) Whale for Straight-cut Origami (large: Figure 26).

FIGURE 1. Two fold-and-cut challenges.

NOTE: For many of the figures in this chapter, the Appendix includes versions large enough for folding and cutting; see Page 105 and following. Whenever such a larger version exist, we include a

reference in the caption of the small figure. Most of these figures were created using the free dynamic geometry software GeoGebra; see Section 4.

1.1. Erik Demaine. We learned about this problem from a 2005 New York Times article about Erik Demaine, a mathematician at the Massachusetts Institute of Technology who –then in his early twenties– has become the leading theoretician in the emerging field of origami mathematics, the formal study of what can be done with a folded sheet of paper. Much of the material in this chapter is based on our attempts to understand the mathematics of the fold-and-cut problem as laid out in a series of papers by Erik Demaine and his collaborators. For more information about these papers and lectures, see the "Further Investigations" in Section 7.

Here is what Erik Demaine tells us about the history of the problem:

> The first published reference to folding and cutting of which we are aware is a Japanese book, Wakoku Chiyekurabe (Mathematical Contests), by **Kan Chu Sen** (; -), published in 1721. This book contains a variety of problems for testing mathematical intelligence. One of the problems asks to fold a rectangular piece of paper flat and make one complete straight cut, so as to make a typical Japanese crest called *sangaibisi*, which translates to "three folded rhombics." [*See Figure 1(a) for a rendition of a* sangabisi *crest.*]
>
> Another early reference to folding and cutting is a July 1873 article National Standards and Emblems in Harper's New Monthly Magazine, volume 47, number 278. This article tells the story about Betsy Ross and her relation to the American flag. It claims that in 1777, George Washington and a committee of the Congress showed Betsy Ross plans for a flag with thirteen six-pointed stars, and asked whether she could make such a flag. She said that she would be willing to try, but suggested that the stars should have five points. To support her argument, she showed how easily such a star could be made, by folding a sheet of paper and making one cut with scissors. The committee decided to accept her changes, and George Washington made a new drawing, which Betsy Ross followed to make the first American flag.
>
> Folding and cutting may have been used for a magic trick by Houdini, before he became a famous escape artist. His 1922 book Paper Magic (E. P. Dutton & Company, pages 176-177) describes a method for making a five-pointed star. According to David Lister, this book appears to have been ghostwritten by another magician, Walter Gibson.
>
> Another magician, Gerald Loe, studied the fold-and-cut idea in some detail; his 1955 book Paper Capers (Magic, Inc., Chicago) describes how to cut out arrangements of various geometric objects, such as a circular chain of stars.
>
> Martin Gardner wrote about the fold-and-cut problem in his famous series in Scientific American (Paper cutting, chapter 5 of New Mathematical Diversions (Revised Edition), Mathematical Association of America, Washington, D.C., 1995.). Gardner was particularly impressed with Loe's ability to cut out any desired letter of the alphabet. He was also the first to state cutting out complex polygons as an open problem. What are the limits of this fold-and-cut process? What polygonal shapes can be cut out?

2. Symmetric Shapes

If you boldly tried to fold the whale shape in order to cut it out with a single straight cut, you may have realized that this is not so easy. Often in mathematics, when we encounter a problem that seems too difficult to tackle with the tools we have at hand, we shift to a somewhat easier problem, in the hope of gaining a deeper understanding of the problem and of learning new tools. Let us therefore start with some shapes that are less complex and that exhibit some symmetry.

FIGURE 2. Flower with line of symmetry.

Consider the shape shown in Figure 2 and the ***line of symmetry*** drawn into it. The left hand side of the shape is (close to) a mirror image of the right hand side of the shape, so that if you fold the shape along this line of symmetry, the two sides will match. We say that the shape has ***reflectional symmetry***; for more on geometry and symmetry, see references [**9, 19, 20**].

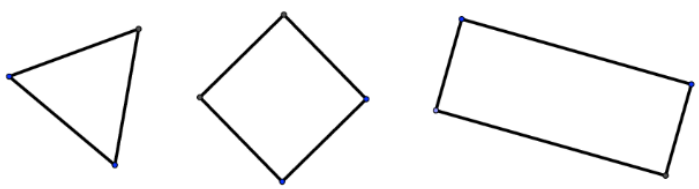


FIGURE 3. Three Geometric Shapes for Straight-cut Origami (large: Figures 14-16).

Figure 3 shows three simple geometric shapes: an equilateral triangle, a square, and a rectangle.

6. Find a way to fold up the equilateral triangle in Figure 3 so that you can cut it out with a single straight cut. Once you succeed, clearly mark on both pieces of paper (the triangle and the outside) the actual fold lines you used. Explain your strategy.

7. As before, find the straight-cut folding for the square. Explain your strategy.

8. Now, find the straight-cut folding for the rectangle. Explain your strategy.

9. Take a new printout of each shape, draw in all the lines of reflection. What do you notice when you compare these with your fold lines? Write down everything you notice.

Figure 4 shows two stars similar to those that Betsy Ross may have worked with in designing the American flag: a five-pointed star and a six-pointed star.

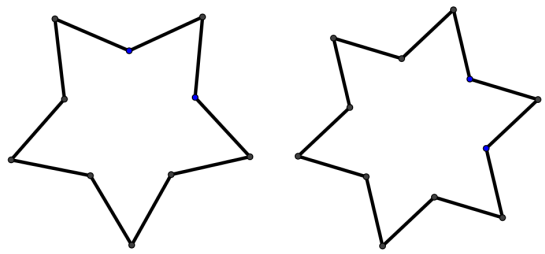


FIGURE 4. Two Star Shapes for Straight-cut Origami (large: Figures 17-18).

10. Find a way to fold up the five-pointed star so that you can cut it out with a single straight cut. Once you succeed, clearly mark on both pieces of paper (the star and the outside) the actual fold lines you used. Explain your strategy.

11. Now, find the straight-cut folding for the six-pointed star. Explain your strategy.

12. Take another five-pointed star and shade the interior. As you fold it up again, make note whether shaded areas get folded onto shaded or unshaded paper (or both). Explain any patterns you see. Would the result have been the same for the six-pointed star?

13. Take a new printout of the six-pointed star, draw in all the lines of reflection and compare them with your fold lines. What do you observe? Would the result have been the same for the five-pointed star?

14. Classroom Discussion: Share your observations about straight-cut folds for symmetric shapes. Do you feel that you would be able to fold and cut any polygon with at least one line of symmetry? Try a few examples to check whether your conjecture is correct.

All the shapes discussed in this section are what are called **polygons**. The word "polygon" derives from the Greek $\pi o\lambda\upsilon\sigma$ (polys) meaning "many" and $\gamma\omega\nu\iota\alpha$ (gōnia), meaning "knee" or "angle". Traditionally, a polygon is a plane figure that is bounded by a closed path or circuit, composed of a finite sequence of straight line segments (i.e., by a closed polygonal chain). These segments are called its **edges** or **sides**, and the points where two edges meet are the polygon's **vertices** or **corners**. An n**-gon** is a polygon with n sides. The interior of the polygon is sometimes called the **body of the polygon**.

A **regular polygon** is a polygon that is **equiangular** (all angles are equal in measure) and **equilateral** (all sides have the same length).

15. When we fold a paper flat and cut along a straight line, will it always separate into two pieces, or can it result in more than two pieces?

16. When we fold a paper flat and cut along a straight line, will the shape(s) you cut out always consist of polygons? Show your thinking and explain your reasoning.

17. Will the resulting shape(s) always be *symmetric*? Explain.

18. Will the shape(s) you cut out always consists of *regular* polygon(s)? Explain.

3. Irregular Shapes

We noticed that lines of symmetry are very helpful in folding polygons so that we can cut them out with a single straight cut. Yet, with shapes that are less symmetric, it is no longer clear how to proceed.

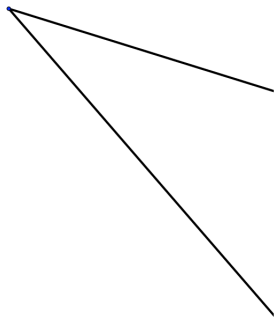


FIGURE 5. Single Angle for Straight-cut Origami (large: Figure 19).

19. Find a way to fold the single angle shape in Figure 5. How does the folding of this shape relate to the work you did with the symmetric shapes in Section 2? Explain.

You may find the following series of investigations more challenging than previous ones. It may take you more than just one attempt with each shape, sometimes *many* more. You may observe that using ideas gained from working with symmetric shapes may not be enough to fold these irregular shapes. Do not be discouraged. Where could you go for some new ideas? Do not discard the results of your attempts. Instead, use them as resources to analyze carefully what the results look like, and why they do not *completely* accomplish the task. Also keep an eye on what happens to interior and exterior areas when folding.

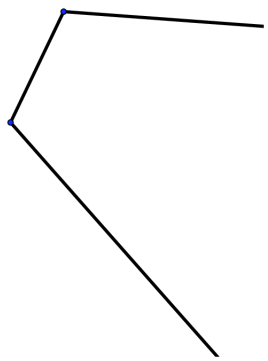


FIGURE 6. Double Angle for Straight-cut Origami (large: Figure 20).

Mathematicians have a name for the line (or line segment) that divides an angle into two equal parts: they call this an **angle bisector**.

20. Find a way to fold the double-angle shape in Figure 6.

21. Find a way to fold the ***irregular triangle*** in Figure 7 (three sides all of different lengths).

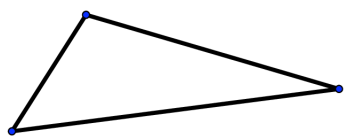


FIGURE 7. An Irregular Triangle for Straight-cut Origami (large: Figure 21).

22. Classroom Discussion: Other than angle bisectors, what other kinds of lines did you use to fold up the shapes in Figure 6 and Figure 7? How would you describe those?

23. An angle bisector is a line of symmetry for one angle. Is it also a line of symmetry of the *entire* shape? What are the consequences of folding along an angle bisector all the way through the shape?

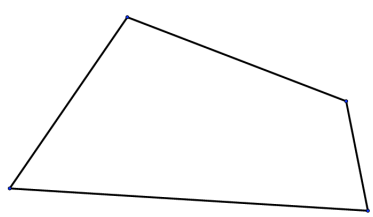


FIGURE 8. An Irregular Quadrilateral for Straight-cut Origami (large: Figure 22).

24. Find a way to fold the ***irregular quadrilateral*** in Figure 8 (four sides all of different lengths).

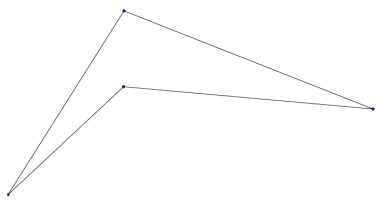
25. Did your observations about folding the double-angle, or the irregular triangle, help you in folding the irregular quadrilateral? Explain.

26. Were there any new ideas that you used for this shape? Explain.

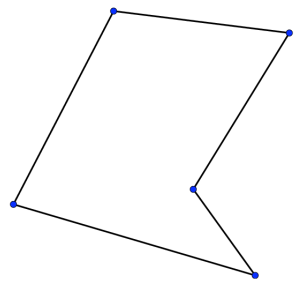
27. Classroom Discussion: Given a line ℓ and a point P not on ℓ, we can construct the **perpendicular**. This is a line through P which makes a $90°$ angle with ℓ. Look back at your folding patterns and find perpendiculars. What do you notice?

28. Next consider the shapes in Figure 9. In what ways are these shapes different from ones you have considered in this section so far? Explain.

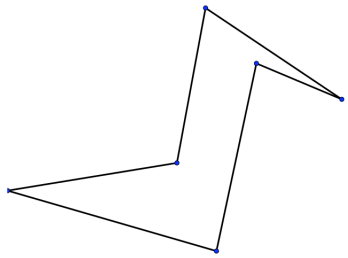
29. Create three geometric shape of your own that belong to this new type. Explain why each belongs to this new type.



(a) Quadrilateral (large: Figure 23).



(b) Pentagon (large: Figure 24).



(c) Hexagon (large: Figure 25).

FIGURE 9. Three Shapes for Straight-Cut Origami.

30. Find a way to fold the quadrilateral in Figure 9(a) so that you could cut it out with a single straight cut.

31. Find a way to fold the pentagon in Figure 9(b) so that you could cut it out with a single straight cut.

32. Find a way to fold the ***non-convex hexagon*** in Figure 9(c) (six sides, segments between some of the vertices fall outside the shape) so that you could cut it out with a single straight cut.

33. Once you know how to fold and cut these shapes, make another folded copy of each but *do not cut it out*. Carefully unfold it, making sure to note which of the fold lines were used in your final version, and which were not. Clearly mark all the fold lines that were needed for your final version.

34. Writing Assignment: Using the marked and labeled shapes you created in Investigation **33** as resources, clearly describe a geometric way in which these folds relate to the original lines of the polygon. Write a complete and careful summary of your observations and findings.

4. Dynamic Explorations

We have now seen how certain geometric lines (angle bisectors and perpendiculars) relate to certain fold lines in straight-cut origami. In this section, you will use the dynamic geometry software called GeoGebra. The ability to dynamically change the shape may allow you to more fully explore the relationship between a shape and its straight-cut fold lines.

35. Download the free GeoGebra software from `http://www.geogebra.org/`. Figure 10 shows a screenshot of a GeoGebra session editing the *sangabisi* crest. The main window contains an image of the shape which you can modify by clicking and dragging. A row of square icons along the top gives access to various geometric tools; hovering over them with the mouse cursor will pop up the names of the tools. A complete list of "free" and "dependent" objects (points, lines, etc.) is shown along the left. An input bar at the bottom accepts direct typed input. Objects may be manipulated using either one of these interfaces.

36. Play around and create a few shapes, using the following tools: New Point, Line Through Two Points, Segment Between Two Points, Perpendicular Line, Angle Bisector, Intersect Two Objects; you may need to click on the little triangles in the bottom right corner of each of the icons in order to see some of these options.

37. Choose the Move tool (first icon on the left) to move some of your lines or points. Explore how to save, export and print your workbook.

38. In order to change basic aspects of your objects, such as the line thickness or color, right-click on the object and choose Object Properties. You can also switch off labels this way. Full documentation is available via Help.

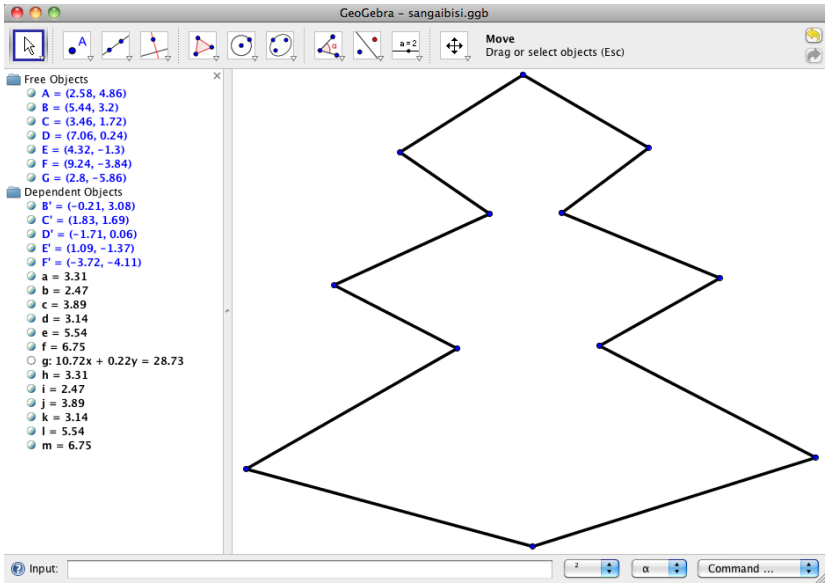


Figure 10. GeoGebra screenshot showing the *sangabisi* crest.

39. Create an irregular quadrilateral similar to that in Figure 8 and draw in the geometric lines that tell you where it needs to be folded for straight-cut origami. Change your quadrilateral using the Move tool and explore how the fold lines change as a result. Write down your observations.

40. Print out your quadrilateral and see whether you can fold it up. Compare whether your printout correctly predicted *all* the fold lines you find. Write down your observations.

41. Independent Investigation: Now create several shapes of your own. Explore how to construct as many of the fold lines as you can. Share your shapes with the class.

5. Further Investigation: Whale

Here is a challenge: can you fold up the whale in Figure 11 so that you can cut it out with a single straight-cut?

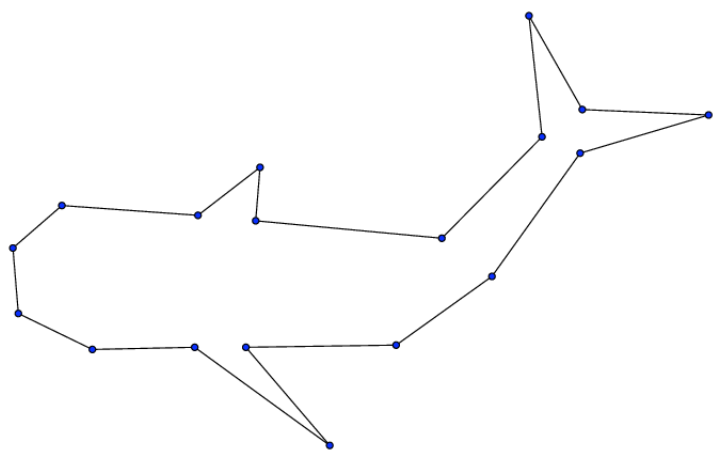


Figure 11. Whale for Straight-cut Origami (large: Figure 29).

As usual in mathematics, if a problem is really difficult, we try to break it down into smaller problems that are hopefully easier to solve. In the following investigations, we first look at three parts of the whale separately: the head, the middle including the fins, and the tail, respectively are shown in Figures 12(a)-12(c). Feel free to use GeoGebra to investigate potential fold lines for these shapes. If you feel bold, skip the separate parts of the whale and jump right in and fold the full whale.

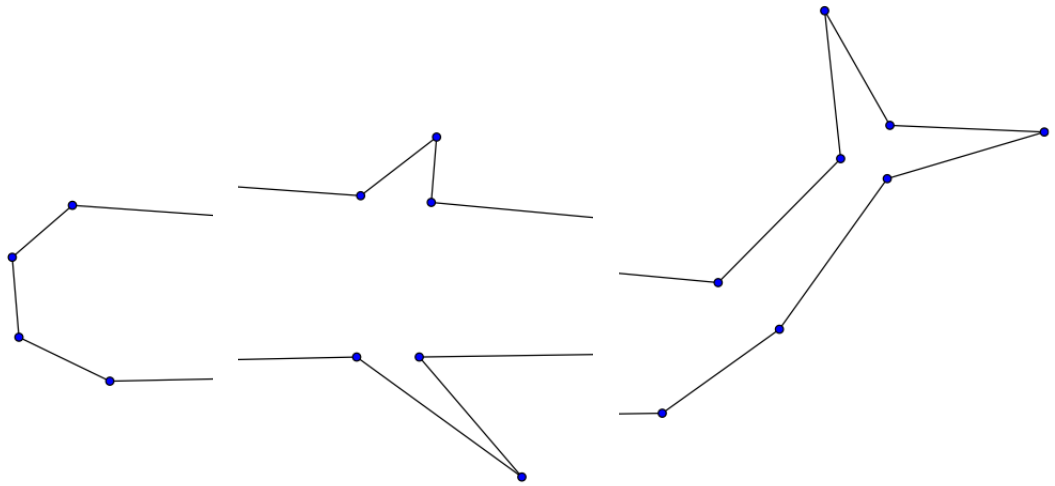
F1. Figure 12(a) shows the head of the whale. Find a way to fold it up so you could cut it out with a single straight cut.

F2. The whale's middle section including the two fins is shown in Figure 12(b). Find a way to fold it up so you could cut it out with a single straight cut.

F3. Now consider the whale's tail in Figure 12(c). Find a way to fold it up so you could cut it out with a single straight cut.

F4. For any of these investigations, did you need to develop any new ideas beyond the ones you've seen before? Explain.

F5. Based on your understanding of the three separate pieces, find a way to fold up the complete whale; see Figure 11.



(a) Whale Head (large: Figure 27).

(b) Whale Middle (large: Figure 28).

(c) Whale Tail (large: Figure 29).

FIGURE 12. Three Parts of the Whale for Straight-Cut Origami.

6. Further Investigation: Eric Demaine's Shapes

Eric Demaine has a few shapes online at

`http://erikdemaine.org/foldcut/examples/`.

F6. Try to fold and cut a few of these. Do as many as you like. Describe how these models relate to your investigations in this chapter.

7. Further Investigation: The Whole Story – Paper and Video

Resources: Erik Demaine's paper "Folding and Cutting Paper" – co-authored with Martin Demaine and Anna Lubiw – explores and explains many of the mathematical ideas behind finding *all* the fold lines; see `http://erikdemaine.org/papers/JCDCG98/paper.pdf`

In addition, you can find a video of a lecture by Erik Demaine online where he talks about this material in more detail; see `http://courses.csail.mit.edu/6.849/fall10/lectures/L07.html`.

F7. Based on what you learn in the paper and the video, describe the following concepts and how they relate: Straight Skeleton Graph; Perpendicular Graph; Corridors. How do these tools and concepts show why (almost) any polygonal shape can be folded and cut out in a single straight cut.

8. Appendix: Large Shapes

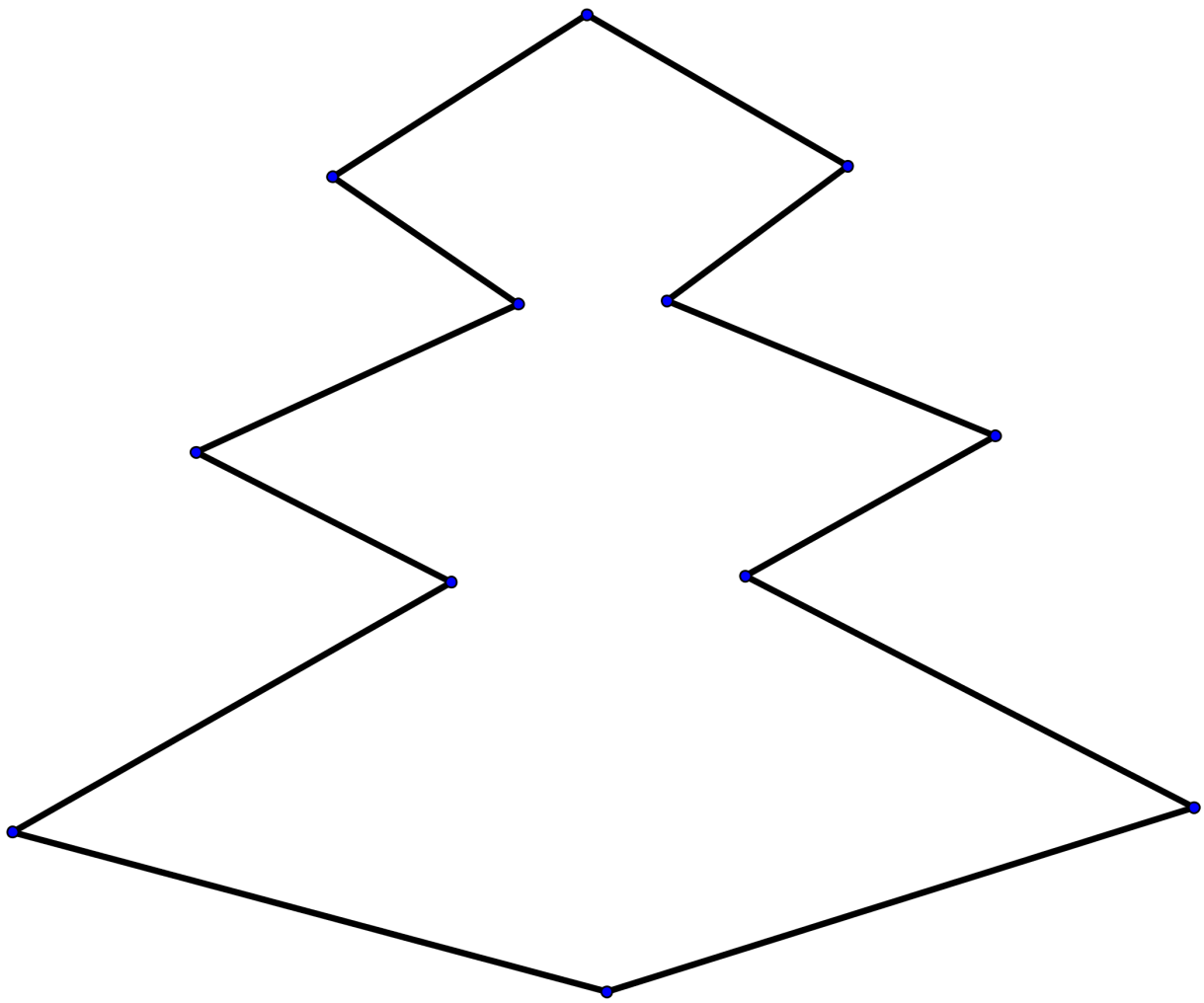


FIGURE 13. Japanese *Sangabisi* crest.

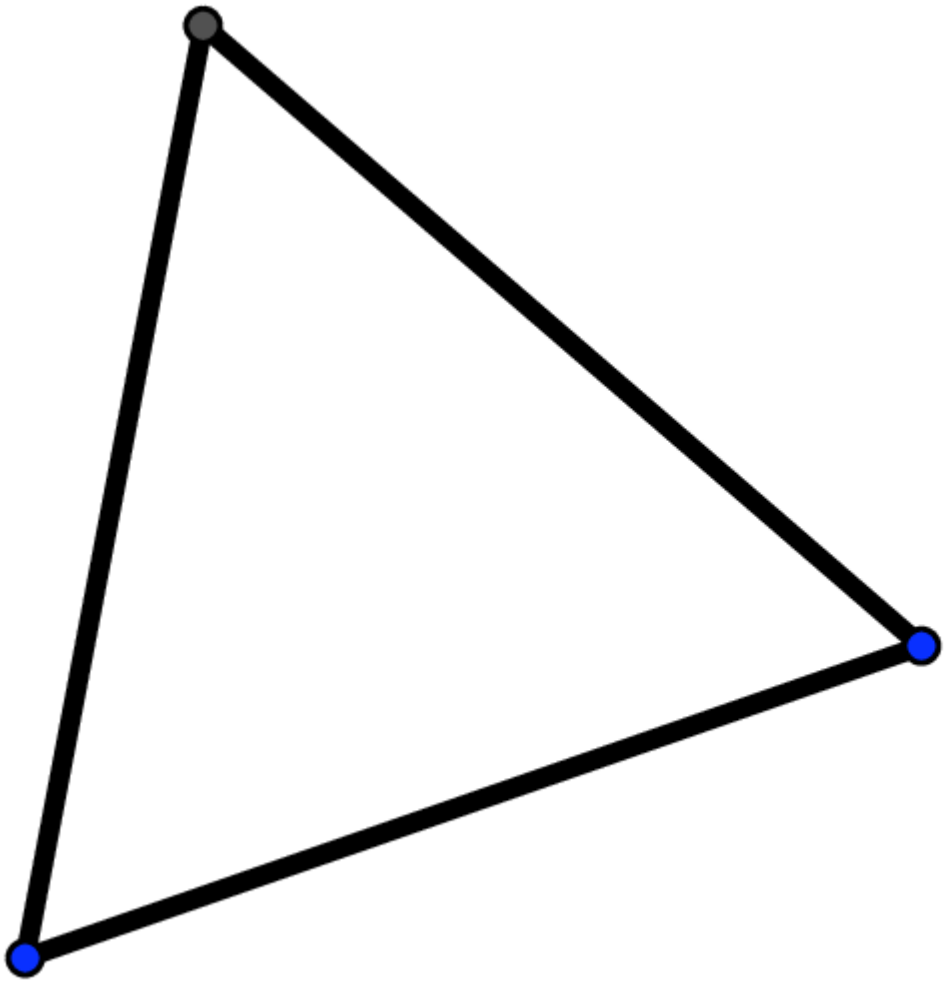


FIGURE 14. Equilateral triangle for Straight-cut Origami.

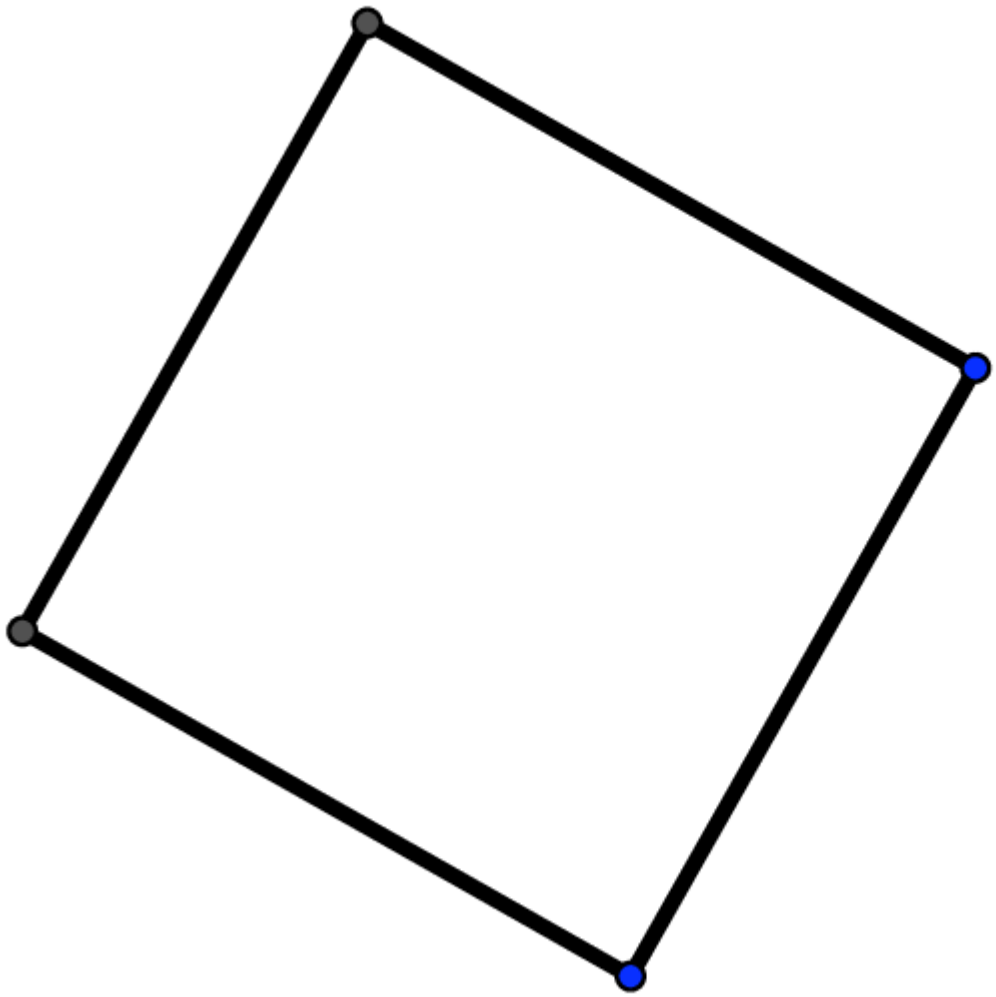


FIGURE 15. Square for Straight-cut Origami.

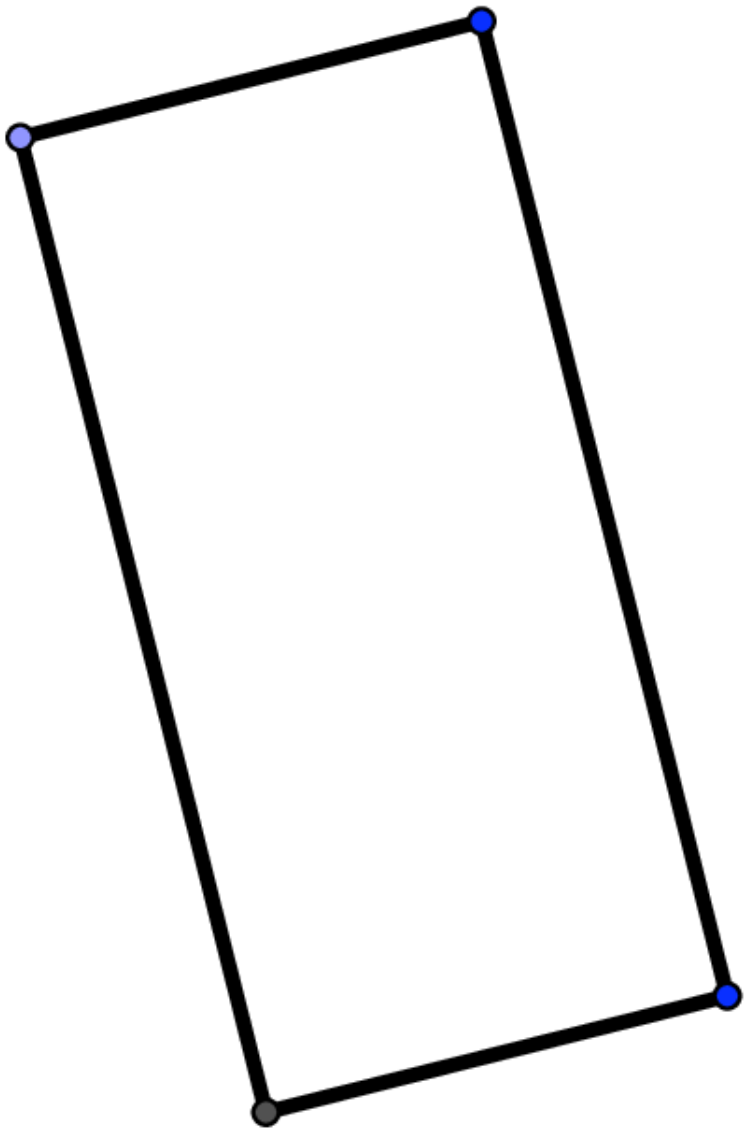


FIGURE 16. Rectangle for Straight-cut Origami.

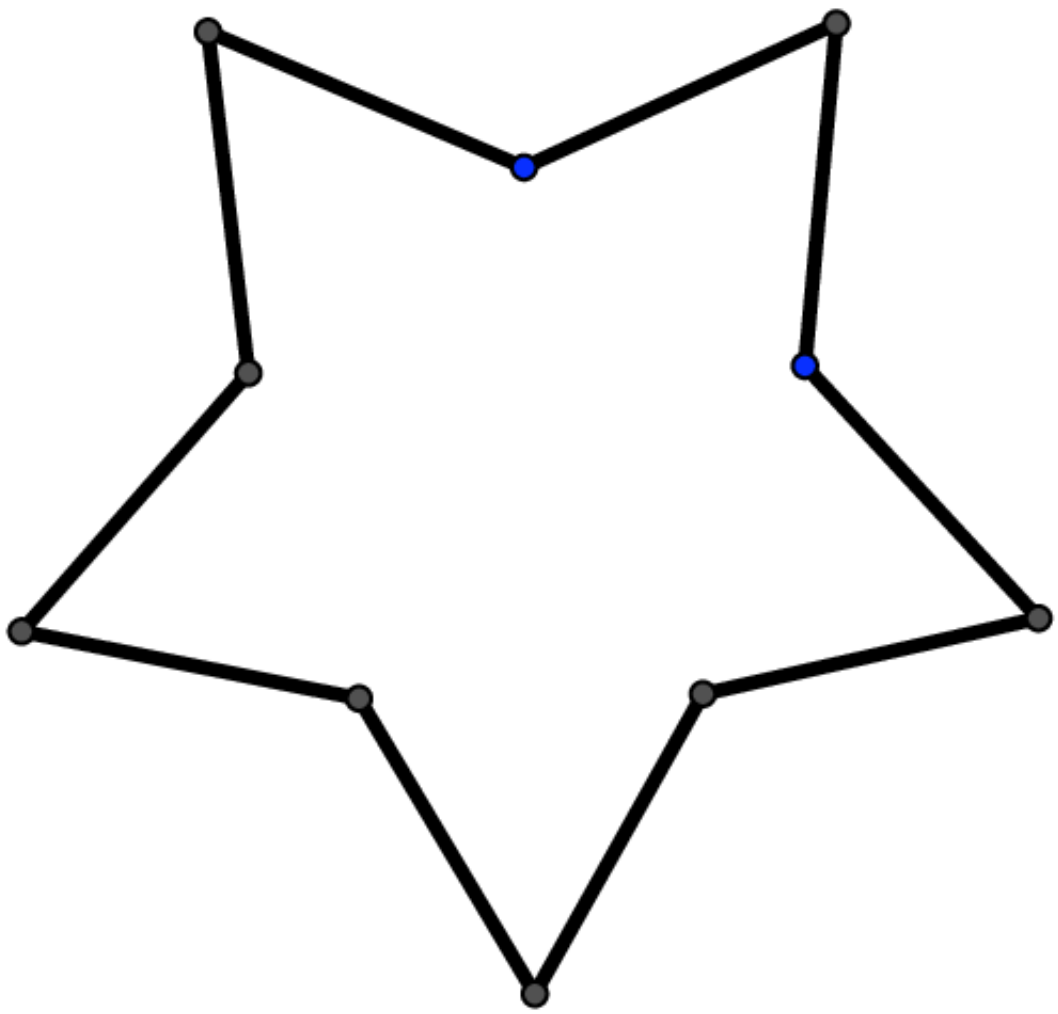


Figure 17. Five-pointed Star for Straight-cut Origami.

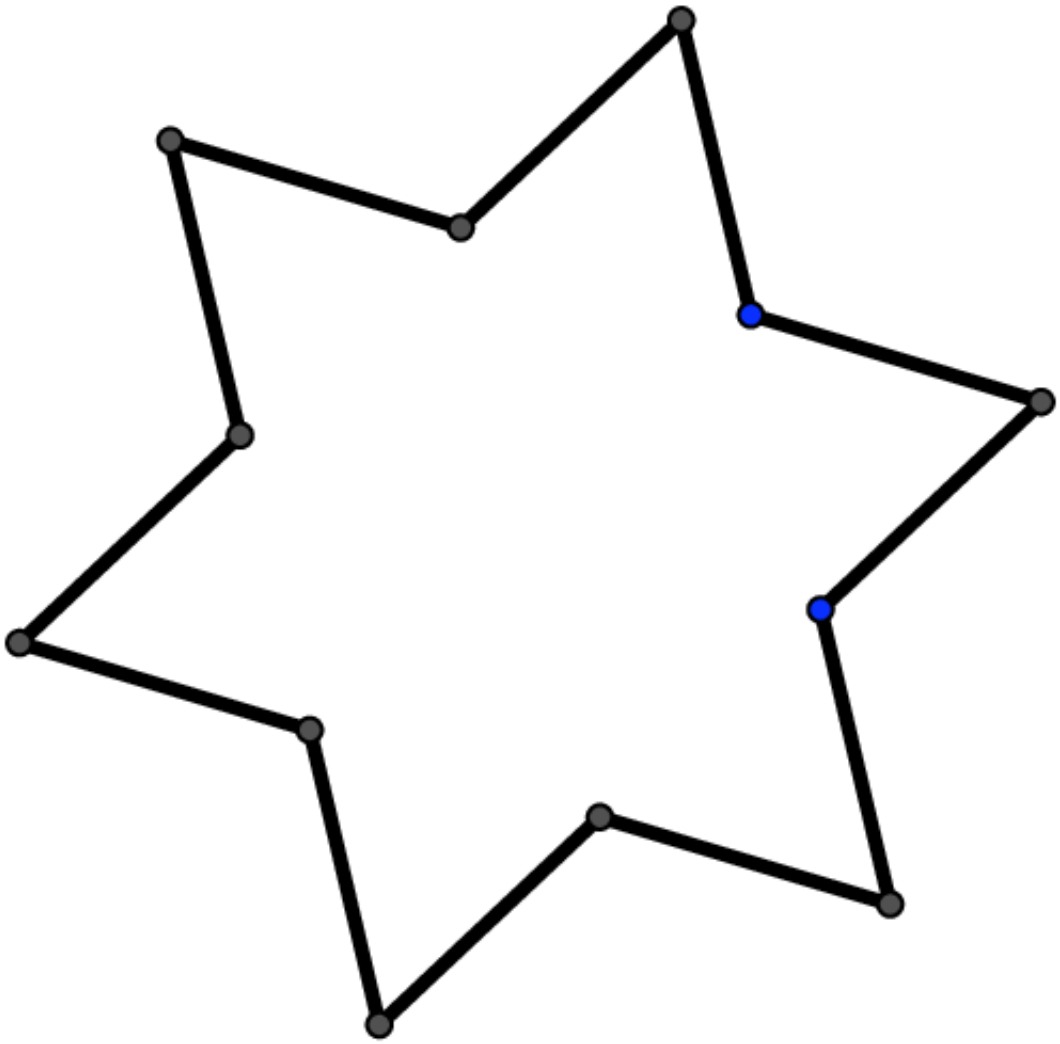


Figure 18. Six-pointed Star for Straight-cut Origami.

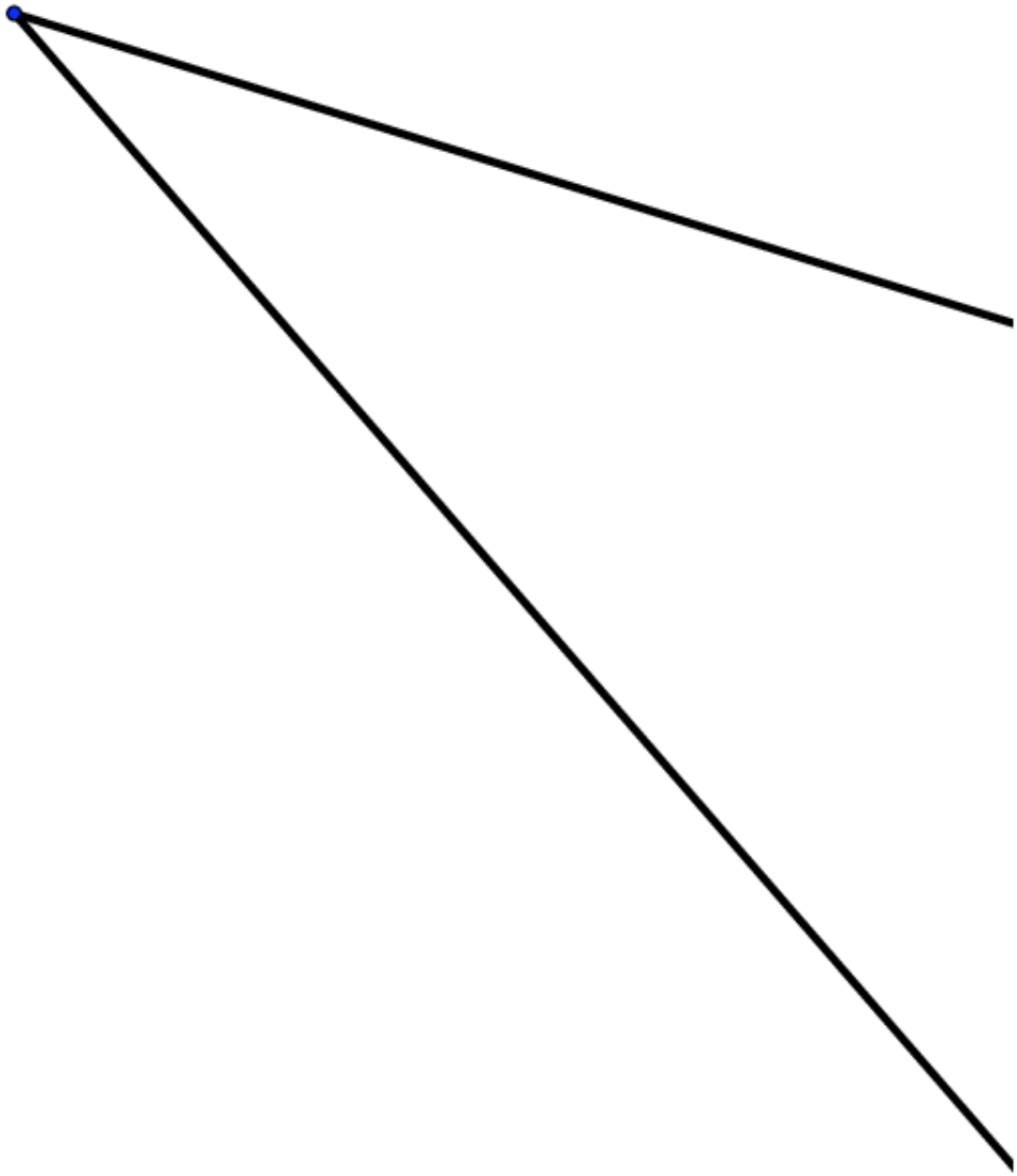


Figure 19. Single Angle for Straight-cut Origami.

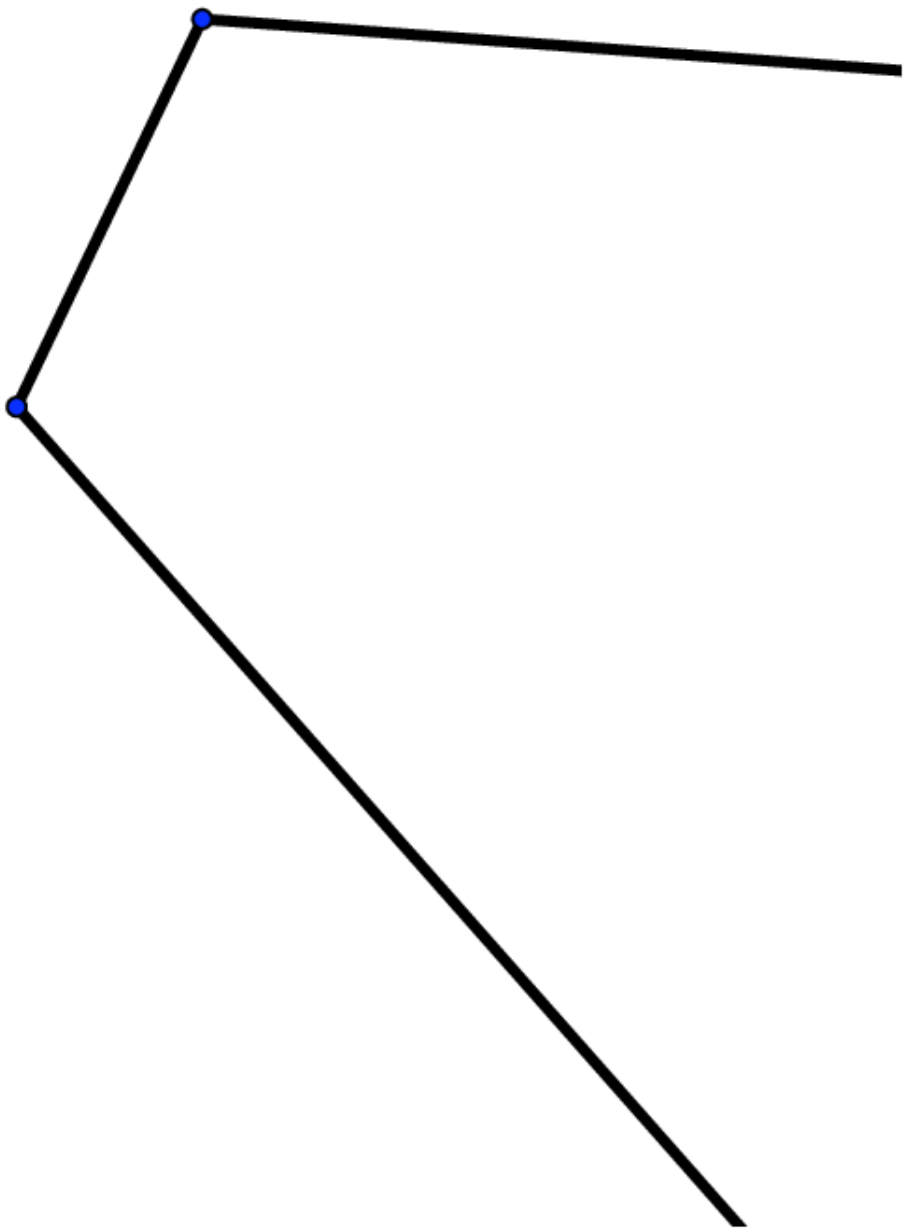


FIGURE 20. Double Angle for Straight-cut Origami.

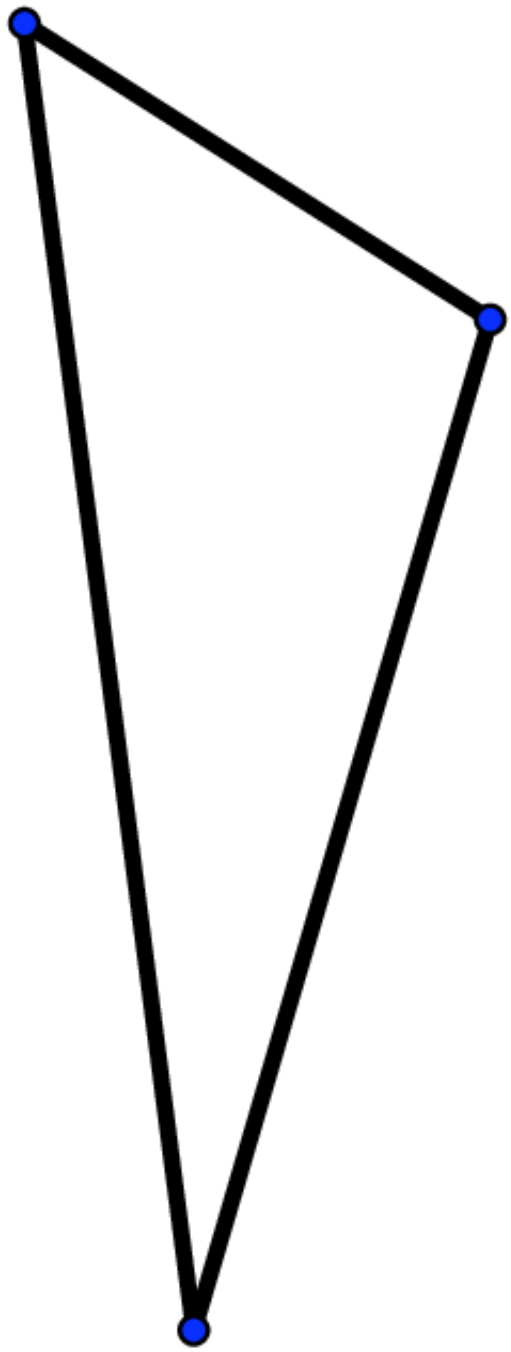


Figure 21. An Irregular Triangle for Straight-cut Origami.

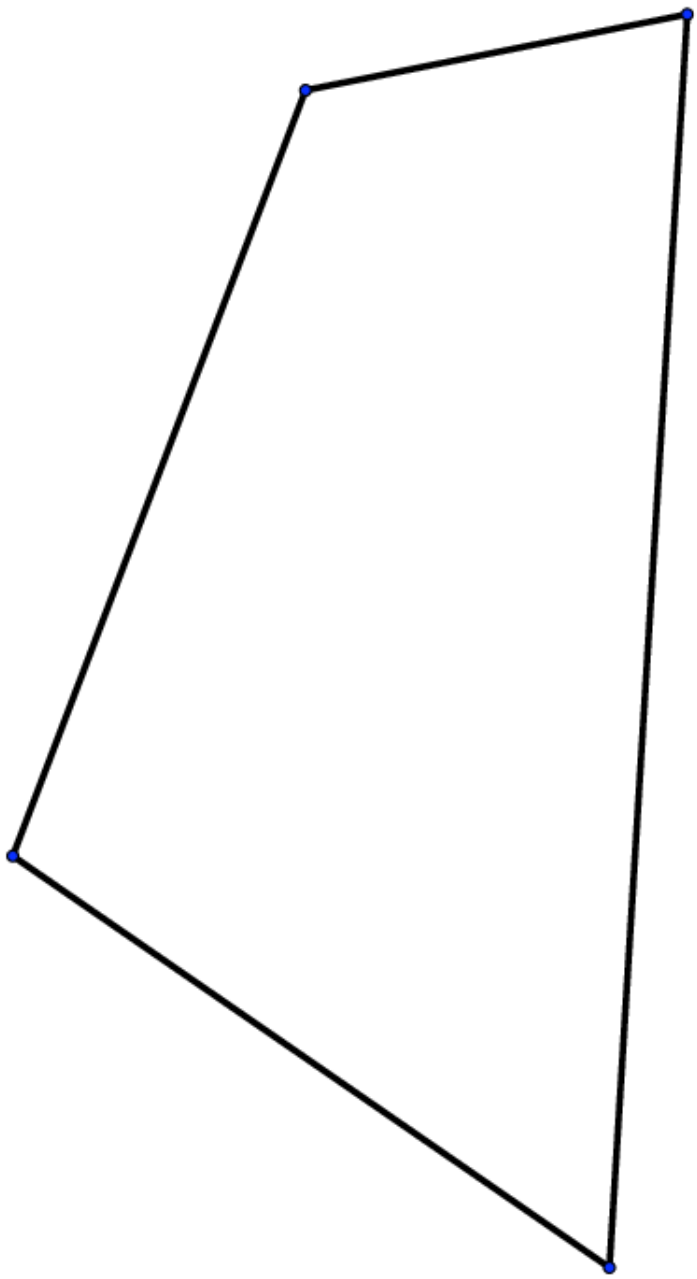


FIGURE 22. An Irregular Quadrilateral for Straight-cut Origami.

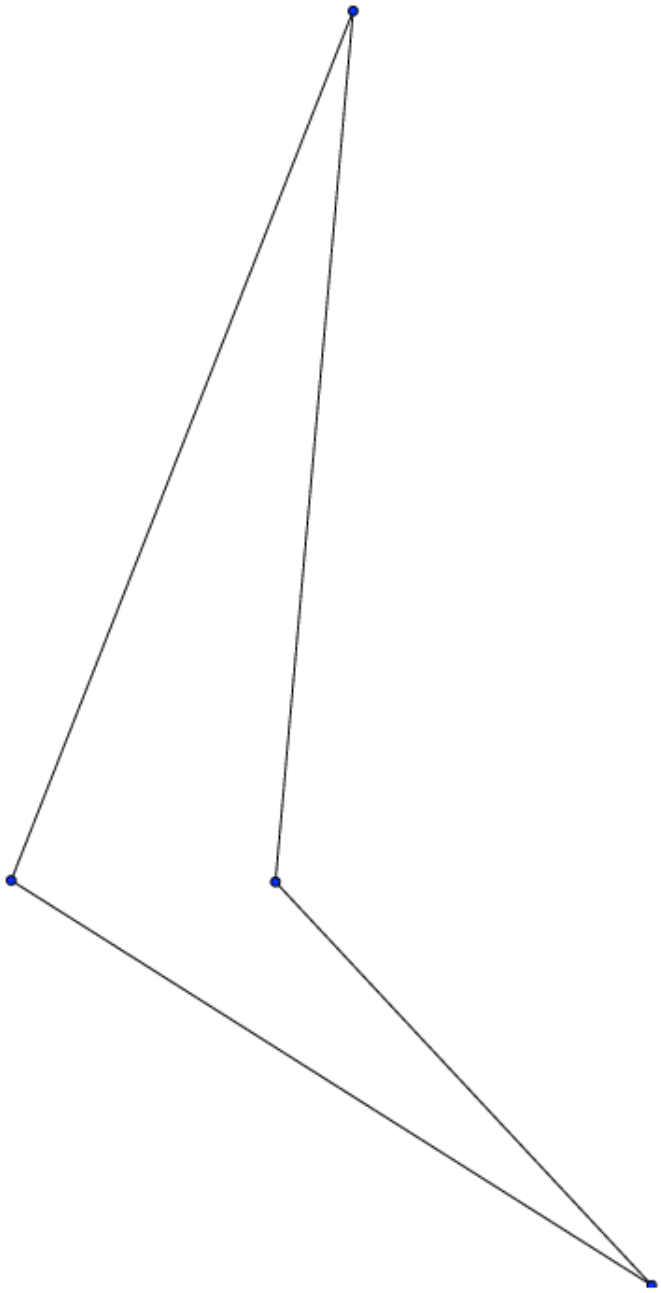


FIGURE 23. An Irregular Quadrilateral for Straight-cut Origami.

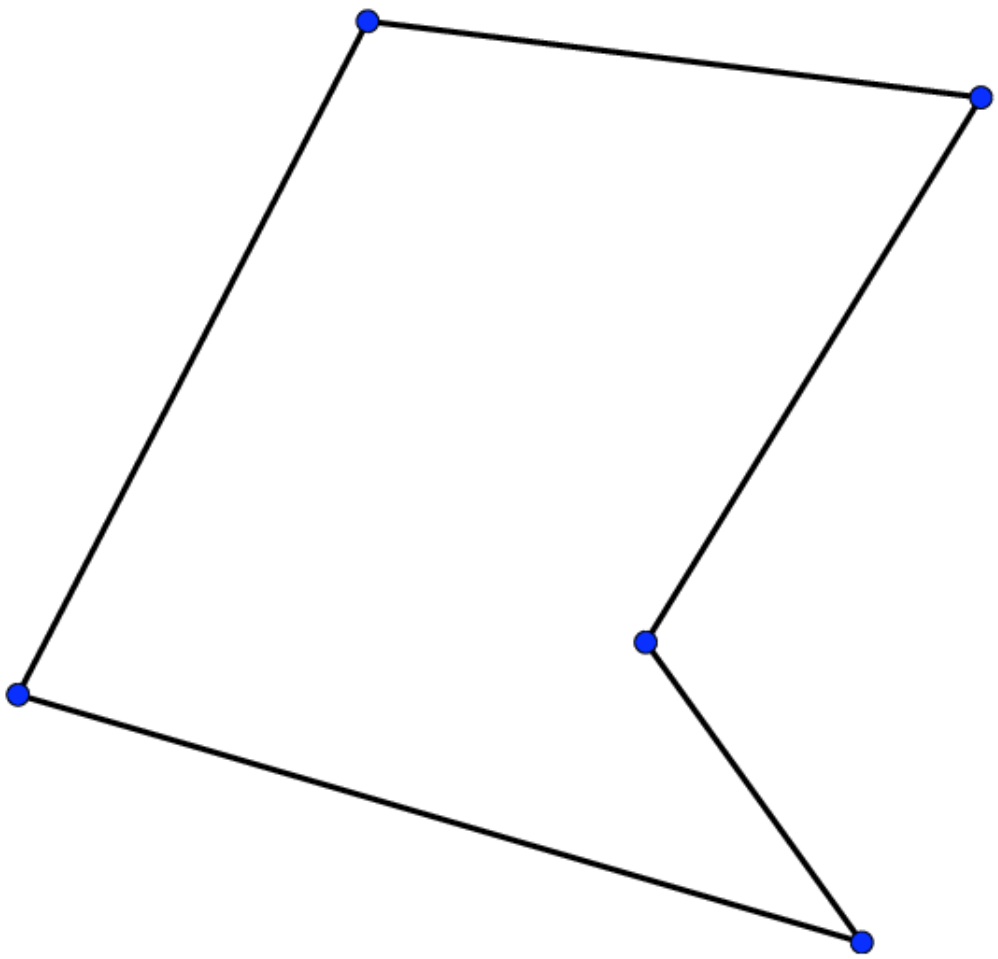


FIGURE 24. An Irregular Pentagon for Straight-cut Origami.

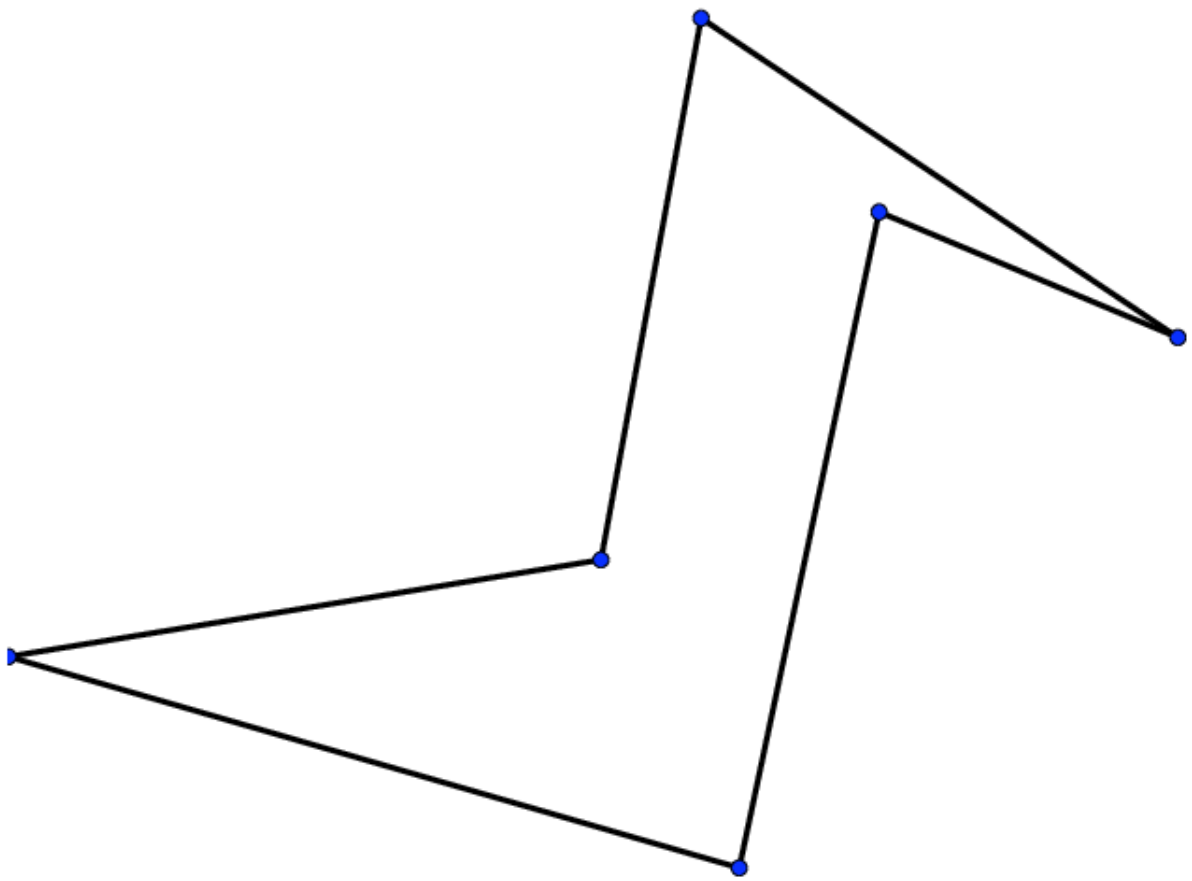


FIGURE 25. An Irregular Hexagon for Straight-cut Origami.

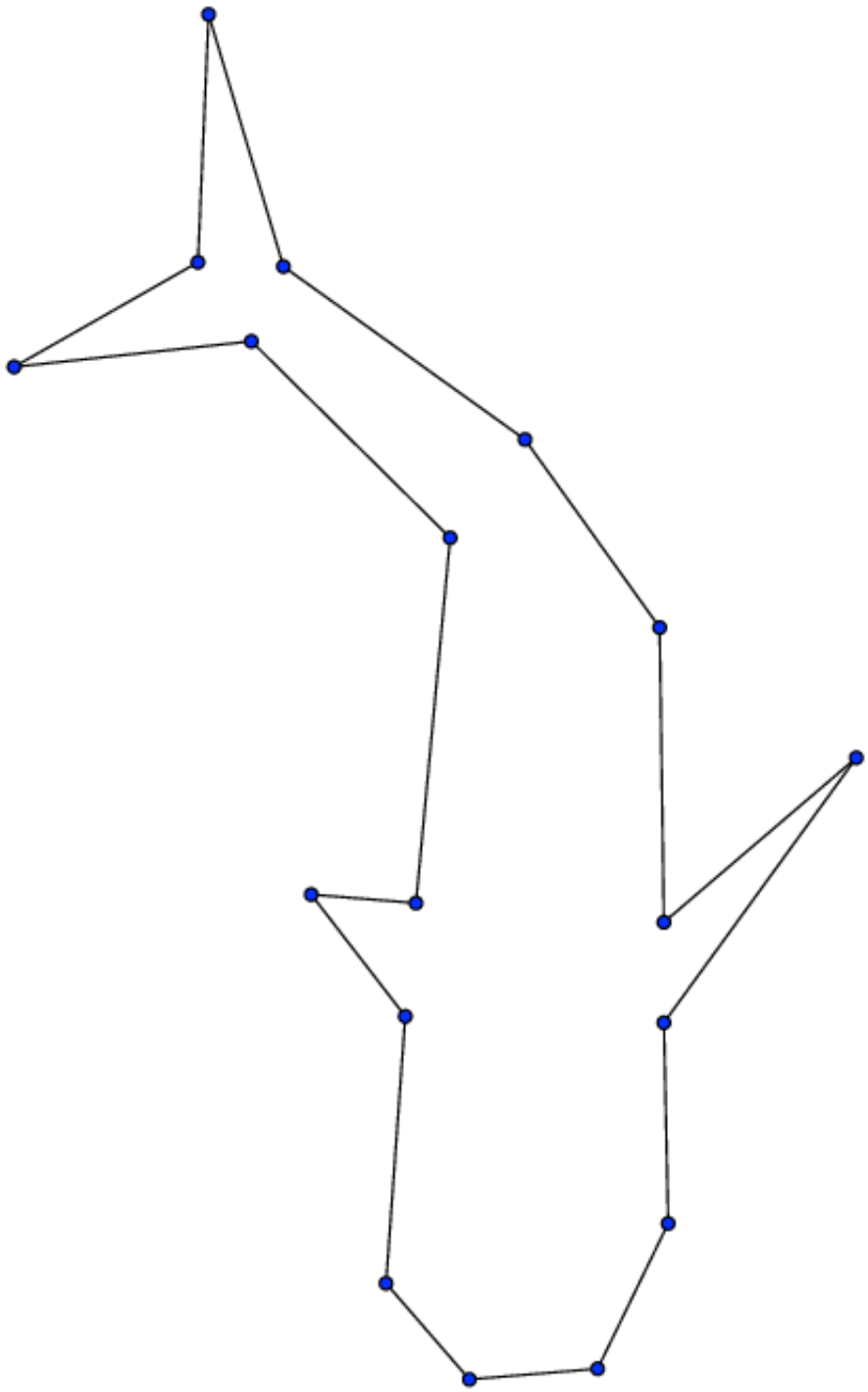


Figure 26. Whale for Straight-cut Origami.

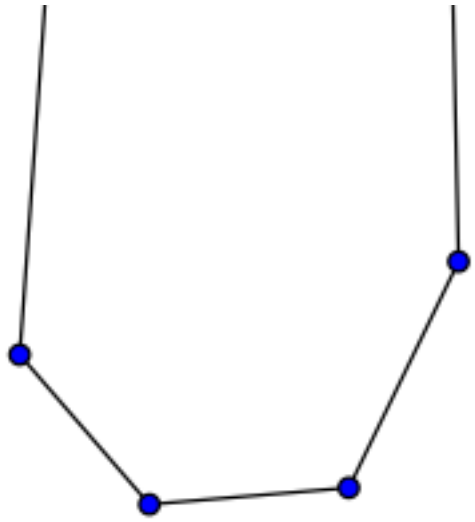


FIGURE 27. Whale Head for Straight-cut Origami.

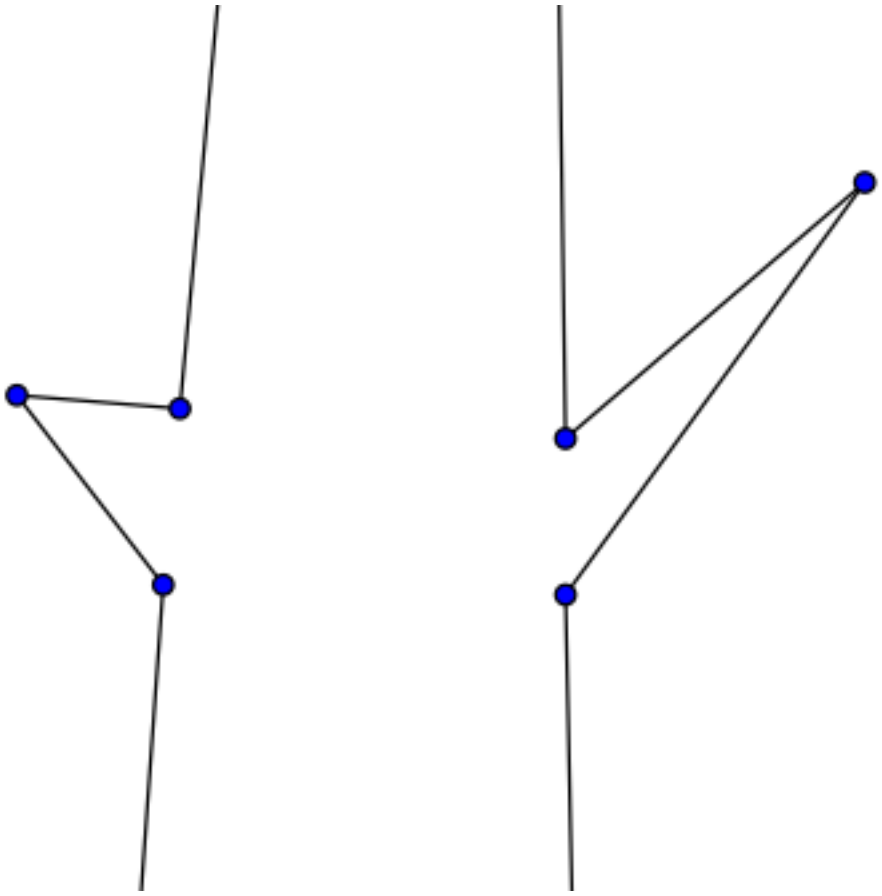


FIGURE 28. Whale Middle for Straight-cut Origami.

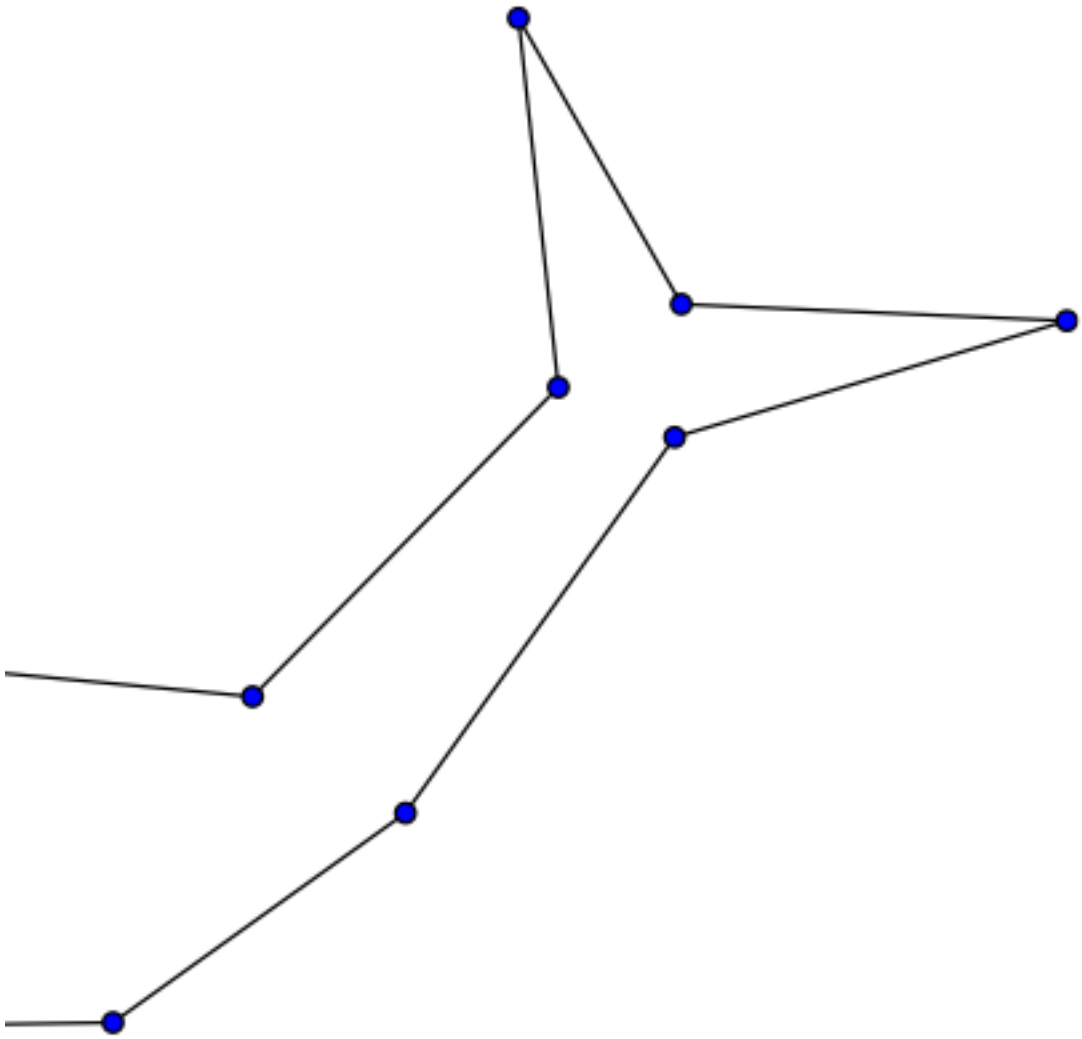


FIGURE 29. Whale Tail for Straight-cut Origami.

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