CHAPTER 4

Rigor and Divergence

> The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds.
>
> **Georg Cantor** (German Mathematician; -)

> Since human beings have never encountered actually infinite collections of things in our material existence, all of our attempts to deal with them must involve projecting our finite experience... Therefore, we must rely on logical reasoning... and then be prepared to accept the consequences of our reasoning, regardless of whether or not they conform to our intuitive feelings.
>
> **William P. Berlinghoff** (; -)
>
> **Kerry E. Grant** (; -)

In the later part of the Chapter 3 you developed a formula for the sum of a geometric series. Namely, the sum is given by

$$1 + r + r^2 + r^3 + ... = \frac{1}{1-r}. \qquad (1)$$

Let's reflect on this for a moment. In Investigations **87-90** in the previous chapter you used this formula to easily determine the sum of four series that previously we had worked fairly hard to compute. Indeed, the sum of *every* geometric series with $-1 < r < 1$ can be computed easily this way. This is a powerful result.

1. Classroom Discussion: It is somewhat typical in mathematics classes for students to be "given" a formula to "apply" to "exercises." Having worked to determine how to deal with a number of different infinite processes and, as part of this, how to determine the sum of an infinite geometric series, you have rediscovered important mathematical results. How does your rediscovery compare with previous experiences you have had in mathematics classes? Was it better to rediscover or would you rather have had us just "give" you this formula to "use?" Explain.

Below you will see how we can use our results on geometric series to help us analyze Zeno's Achilles paradox.

Such success with the infinite is encouraging. Mathematicians of the seventeenth and eighteenth century were buoyed with similar success. The newly discovered calculus, which relied heavily on the notion of the ***infinitesimal***, or infinitely small, and infinite series allowed mathematicians (who were usually scientists as well) to make remarkable progress in the description of the physical universe.

However, as indicated at the end of the previous chapter, there is an important need for rigor. Significant problems associated with the cavalier treatment of the infinite began to be noticed at the outset of the nineteenth century. **David Bressoud** (American Mathematician; -) ascribes the precipitating event to manuscript "Theory of the propogation of heat in solid bodies" by **Jean Baptiste Joseph Fourier** (French Physicist and Mathematician; 1768 - 1830). Bressoud describes the crisis as follows:

> The crisis struck four days before Christmas 1807. The edifice of calculus was shaken to its foundations The nineteenth century would see ever expanding investigations into the assumptions of the calculus, an inspection and refitting of the structure from the footings to the pinnacle, so thorough a reconstruction that the

calculus would be given a new name: *analysis*. Few of those who witnessed the incident of 1807 would have recognized mathematics as it stood 100 years later.[1]

Below you will investigate some of the resulting rigor that was introduced in this revolution. There are a number of troubling paradoxes that remain. But these are now results of the limits of our intuition regarding the infinite, not limits of our understanding of the mathematics surrounding the infinite.

1. Zeno Redux

As in the Chapter 2 we will assume Zeno's hypothetical is 100 meters and that Achilles allows the Tortoise a 50 meter lead. When the race started, Achilles needed to reach the 50m point where Tortoise started. When he reached that point, Tortoise had moved further ahead. We can think of this as the first stage of the race. In the second stage Achilles needed to reach Tortoise's new location. When he reached that point Tortoise again had moved further ahead. These stages continue indefinitely, suggesting that Achilles can never catch Tortoise. This is the paradox.

This is a qualitative analysis. We would like to use what we have learned about series to provide a quantitative analysis. Because this paradox involves both distance and time, speed must be involved. So let us assume that Achilles runs the race at a constant speed of 10 m/sec and Tortoise crawls at a constant speed of 4 m/sec.

2. How long does it take for Achilles to reach the point where Tortoise started the race?
3. Since the beginning gun which started the race, how much time has elapsed?
4. When Achilles reaches the starting point of Tortoise, how much further ahead has Tortoise moved?
5. At this point, how far is Tortoise from the start?
6. How long does it take for Achilles to go from Tortoise's starting location to Tortoise's location at the end of Stage 1 that you found in Investigation **5**?
7. Since the beginning gun which started the race, how much time has elapsed?
8. When Achilles reaches the Tortoise's location at the end of Stage 1, how much further ahead has Tortoise moved?
9. At this point, how far is Tortoise from the start?

We need to continue to identify the times and locations involved in the motions of our two racers. One way to do this is with a table.

Stage	Time Interval	Total Elapsed Time	Distance Covered by Tortoise	Tortoise Location	Distance Covered by Achilles	Achilles Location
0	0	0	0	50	0	0
1					50	50
2						

10. Use your data from Investigations **2-9** to fill in the rows for the first and second stages.
11. Now determine data which enables you to complete the table for stages 3 - 10. Describe any patterns that you see.

[1]From A Radical Approach to Real Analysis, p. 1.

You should notice that both the total elapsed time and the locations of our two runners seem to be *converging* to fixed *limits* as the stage number $n \to \infty$. We would like to determine what these limits are.

12. Using your data, roughly estimate the limits as $n \to \infty$ of the total elapsed time, Tortoise's Location, and Achilles' Location.
13. Write the limit of the Total Elapsed Time as the sum of an infinite series.
14. Use what you have learned about infinite series to determine the sum of this infinite series. Does this limit agree with your estimate above?
15. Write the limit of Tortoise's location as the sum of an infinite series.
16. Use what you have learned about infinite series to determine the sum of this infinite series. Does this limit agree with your estimate above?
17. Check that your previous answer makes sense by using the time in Investigation **13** and the constant rate of speed Tortoise travels.
18. Write the limit of the Achilles's location as ths sum of an infinite series.
19. Use what you have learned about infinite series to determine the sum of this infinite series. Does this limit agree with your estimate above?
20. Check that your previous answer makes sense by using the time in Investigation **13** and the constant rate of speed Achilles travels.
21. Based on these investigations, what happens at the time in Investigation **13**?
22. What happens after the time in Investigation **13**?

Another useful way to envision Zeno's Achilles paradox is to consider it graphically.

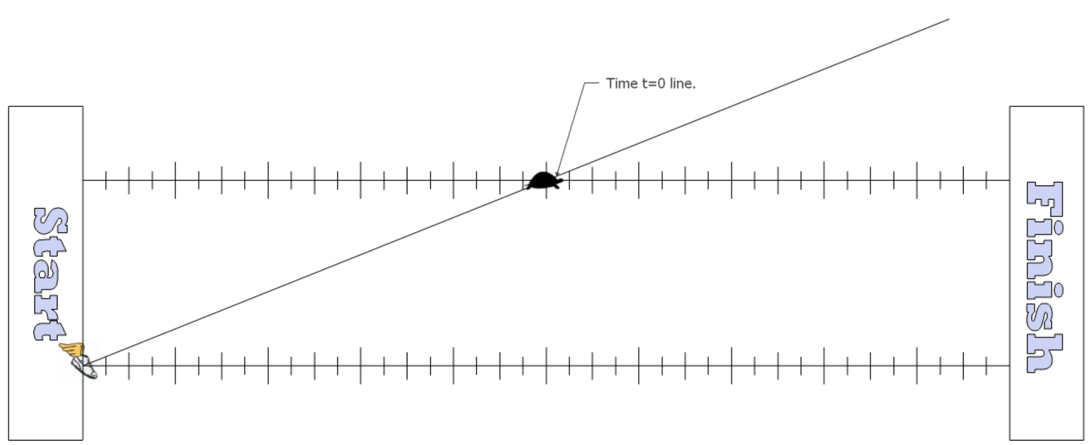

FIGURE 1. Zeno's race between Achilles and Tortoise.

23. An enlarged image of Figure 1 is included in the Appendix. On it, plot the location of Achilles and Tortoise at the end of stages 1 - 6. Keep each racer in their own lane. As has been done for time $t = 0$, draw a line from Achilles' location extending well through Tortoise's location.
24. What do you notice about all of the lines that you have drawn?
25. Your completed figure should now remind you of a *perspective drawing*. It may also remind you of Investigation **58** from the chapter which includes our analysis of the Wheel of Aristotle. Does this figure shed any additional light on Zeno's Achilles paradox? Explain.

In 1901 **Bertrand Russell** (English mathematician, logician, and philosopher; -) remarked:

> Zeno was concerned with three problems...These are the problem of the infinitesimal, the infinite, and continuity...From his to our own day, the finest intellects of each generation in turn attacked these problems, but achieved broadly speaking, nothing...Weierstrass, Dedekind, and Cantor,...have completely solved them. Their solutions...are so clear as to leave no longer the slightest doubt or difficulty. This achievement is probably the greatest of which the age can boast...The problem of the infinitesimal was solved by Weierstrass, the solution of the other two was begun by Dedekind and definitely accomplished by Cantor.

Nonetheless, dissenters remain. In a September, 1994 *Scientific American* feature "Resolving Zeno's Paradoxes", **William I. McLauglin** (; -) stated:

> At last, using a formulation of calculus that was developed in just the past decade or so, it is possible to resolve Zeno's paradoxes. The resolution depends on the concept of infinitesimals, known since ancient times but until recently viewed by many thinkers with skepticism.

At last? What does this mean?

26. Independent Investigation: Interview a mathematicians and report on her/his view of the status of legitimate mathematical solutions to Zeno's paradoxes.

An issue that further complicates Zeno's paradoxes is the distinction between idealized, mathematical models of time and space on the one hand and the real time and space we inhabit on the other. Idealized time and space are continuous and infinitely divisible. However, in prevailing physical theories matter is neither continuous not infinitely divisible. That suggests to some that real space and time are not either.

27. Independent Investigation: Interview a mathematician, philosopher, and/or physical scientist and report on her/his views on the status of legitimate, real world solutions to Zeno's paradoxes.

2. Divergent Series

As authors we have been very careful to include only infinite series that are *well-behaved*. What this means must now be determined, but, up to this point, all of the infinite series, except those in final section about Anna Mills' . . . 999.0, we have considered are *absolutely convergent*.

We now work to clarify what it means for infinite series to be well-behaved. This is an absolutely critical matter, as realized by the famous, but tragically short-lived,[2] **Neils Henrik Abel** (Norwegian Mathematician; 1802 - 1829):

> If you disregard the simplest cases, there is in all of mathematics not a single infinite series whose sum has been rigorously determined. In other words, the most important parts of mathematics stand without a foundation.

28. Consider the sequence $(-1), (-1)^2, (-1)^3, (-1)^4, \ldots$ Simplify the terms in this sequence so the sequence can be written without using exponents.

29. Let $r = -1$. With the help of the ideas in the previous investigation, express the geometric series $1 + r + r^2 + r^3 + \ldots$ without the use of exponents.

[2]Abel made brilliant and insightful contributions to modern mathematics, but, sadly, died of tuberculosis at the age of 27. He is a revered figure in Norway and the Abel Prize is one of the more significant awards in the field of mathematics.

The infinite series in Investigation **29** is called the **Grandi series** after **Guido Grandi** (Italian Mathematician and Jesuit Philosopher; 1671 - 1742).

30. What do you think the sum of the Grandi series is? Explain.

31. Using parentheses, group the first two terms in the series together, the next two terms together, the next two terms after that together, etc. What value does this suggest the sum of the Grandi series will be?

32. Suppose now that you left the first term by itself and instead used parentheses to group the second and third terms together, the fourth and fifth terms together, etc. What value does this suggest the sum of the Grandi series will be?

33. Are your results in Investigations Investigation **31** and Investigation **32** compatible? Explain.

34. The situation in Investigation **33** was described by Grandi as "comparable to the mysteries of Christianity... paralleling the creation of the world out of nothing." What do you think of Grandi's description?

35. Because the Grandi series arose from a geometric series it seems reasonable to use our formula above to ascertain a value for the sum of the series. What value does the formula predict?

36. Classroom Discussion: What do these results suggest about the numbers 0, $\frac{1}{2}$, and 1? Is any of this reasonable? What do you think is an appropriate value for the sum of the Grandi series?

Even the most gifted mathematicians were troubled by paradoxes like the Grandi series. **Leonard Euler** (Swiss Mathematician; 1707 - 1783), one of the greatest mathematicians of all times, held firmly to the conviction that 1/2 was the proper sum of the Grandi series.

The contemporary practice is to name infinite series for which an appropriate sum cannot be found as a *divergent series.*

A very important infinite series whose sum can be rigorously determined is called the **harmonic series**. It is the following infinite series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

37. Write out the first fifteen terms in the harmonic series.

38. Sum the first three terms in the harmonic series.

39. Sum the first seven terms in the harmonic series.

40. Sum the first fifteen terms in the harmonic series.

41. How fast does the sum appear to be growing as you add more terms.

42. Is it clear whether the series will sum to a specific value? If so, what is this value?

Consider now the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \dots \tag{2}$$

43. How many terms of $\frac{1}{2}$ are there?

44. What is the sum of these "$\frac{1}{2}$ terms"?

45. How many terms of $\frac{1}{4}$ are there?

46. What is the sum of these "$\frac{1}{4}$ terms"?

47. How many terms of $\frac{1}{8}$ are there?

48. What is the sum of these "$\frac{1}{8}$ terms"?

49. Do the preceding problems suggest how many terms of $\frac{1}{16}$ there will be in this series. How many?

50. What is the sum of these "$\frac{1}{16}$ terms"?

51. If we sum this series through the end of the "$\frac{1}{4}$ terms", what will the sum be?

52. If we sum this series through the end of the "$\frac{1}{8}$ terms", what will the sum be?

53. If we sum this series through the end of the "$\frac{1}{16}$ terms", what will the sum be?
54. If we sum this series through the end of the "$\frac{1}{32}$ terms", what will the sum be?
55. The sum of this series is greater than 10. Through what terms would we have to add to reach a sum greater than 10?
56. The sum of this series is greater than 100. Through what terms would we have to add to reach a sum greater than 10?
57. The sum of this series is greater than 1,000,000. Through what terms would we have to add to reach a sum greater than 10?
58. Compare the first term of the harmonic series with the first term of this new series. Then compare the second terms of these series. And the third. And the fourth. Whenever they are not equal, which series has the larger terms?

59. Classroom Discussion: Explain how to use the Investigations above to determine precise sums for both the series in (2) and the harmonic series.

In addition to its use for series like the Grandi series, mathematicians also call series whose sums approach $\pm\infty$ *divergent series*.

The proof of the divergence of the harmonic series you just completed is due to **Nicole Oresme** (French Philosopher, Theologian, Mathematician, and Astronomer; c. 1323 - 1382).

3. Devilish Series

> The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes.
> **Neils Henrik Abel** (Norwegian Mathematician; 1802 - 1829)

There is no reason that infinite series must have all positive terms. For example, the **alternating harmonic series** is built from the harmonic series but with every other term negative:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

Shown in Figure 2 are the results of adding a small number of terms of this series. For each of the following, sum the indicated number of terms and then graph the *partial sums* on the graph provided in the appendix:

60. ...the first four terms.
61. ...the first five terms.
62. ...the first six terms.
63. ...the first seven terms.
64. ...the first eight terms.
65. ...the first nine terms.
66. ...the first ten terms.
67. ...the first eleven terms.
68. ...the first twelve terms.

Some of you might have access to technology (e.g. graphing calculators, programming languages, or spreadsheets) that will enable you to easily add dozens, hundreds, thousands, or even millions of terms. Here are a few examples, each correct to 20 decimal places:

Partial Series	Partial Sum
$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{100}$	0.68817217931019520324
$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{500}$	0.69214818055794532542
$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{1000}$	0.69264743055982030967
$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{5000}$	0.69304719055994510942

69. Classroom Discussion: Do the patterns in your partial sums allow you to determine whether a partial sum over- or under-estimates the actual sum? Can you determine a bound on how much

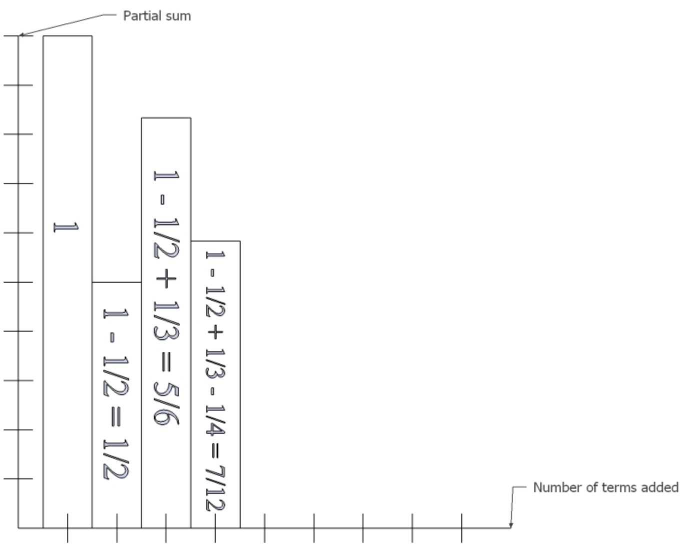

FIGURE 2. Partial sums of the alternating harmonic series.

your partial sums over- and under-estimate the actual sum? How can these observations be combined with your data to find the sum of the alternating harmonic series correct to many decimal places? Do so.

The *natural logarithm* is one of the more important functions in mathematics. Its values can be approximated on most calculators - you use the *ln* function button. The value of $ln(2)$, correct to 9 decimal places, is 0.6931471806. That this is the sum of the alternating harmonic series was discovered in ?? by ??.

70. Is $2 + 3 = 3 + 2$? Why?
71. Is $219 + 3,427 + 5,392 = 5,392 + 219 + 3,427$? Why?
72. Is $12 + 3,492 + 12,987 + 438 + 29,351 + 423 = 438 + 12 + 423 + 12,987 + 29,351 + 3,492$? Why?
73. Does the sum of finitely many terms remain the same when the order of the terms are rearranged?
74. Do you think the sum of infinitely many terms remains the same when the order of the terms are rearranged?

When we rearrange the order in which we add terms in a series, finite or infinite, we call this a ***rearrangement*** - no surprise there. Let us investigate what happens when we rearrange the alternating harmonic series.

75. What would the terms in the next three sets of parenthesis in the series
$$1 - \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{8} - \frac{1}{10}\right) + \dots$$
be?
76. Explain why the infinite series in Investigation **75** is a rearrangement of the alternating harmonic series.
77. Simplify and reduce the terms in each of the five sets of parenthesis in Investigation **75**, leaving your results as fractions.
78. Show that each of the results in Investigation **77** is given by $\frac{-1}{4n(2n+1)}$ for appropriate values of n. (Note: With a little algebra, this result can be proved to continue indefinitely.)
79. What can you conclude about the sum of the infinite series in Investigation **75**? Does this result agree with the sum of the series prior to its rearrangment?

Consider now the infinite series

$$1 + \left(-\frac{1}{2} + \frac{1}{3} + \frac{1}{5}\right) + \left(-\frac{1}{4} + \frac{1}{7} + \frac{1}{9}\right) + \ldots$$

80. What would the terms in the next three sets of parenthesis in this series be?
81. Explain why this infinite series is a rearrangement of the alternating harmonic series.
82. Simplify and reduce the terms in each of the five sets of parenthesis, leaving your results as fractions.
83. Show that each of the results is given by $\frac{-1}{2n(4n-1)(4n+1)}$.
84. What can you conclude about the sum of this infinite series? Does this result agree with the sum of the series prior to its rearrangment?

We have now shown that the alternating harmonic series has *three different sums* when it is rearranged! These are not fictitous sums like those of the Grandi series. These are real sums whose exact values can be rigorously determined. And this shock is just the beginning. For the alternating harmonic series can be rearranged to have *any* sum! This is the striking result of **Georg Friedrich Bernhard Riemann** (German mathematician; 1826 - 1866):

Theorem 4. *(Riemann's Rearrangement Theorem) Suppose that an infinite series of both positive and negative terms converges while the series formed by making all of the original series' terms positive diverges. (Such a series is called* **conditionally convergent***.) Let T be any real number, $+\infty$ or $-\infty$. Then there is a rearrangement of the series that converges to T.*

This is a remarkable theorem which highlights the limits of our finite intuitions in dealing with the infinite. Unlike some of the earlier surprises and paradoxes, we are not seeing the faults in our reasoning. We've illustrated this result with rigor, the theorem itself is established deductively in a fairly straightforward way (see Further Investigations), and we must accept this result as a wonderful glimpse into the world of the infinite.

4. Gabriel's Wedding Cake

In this section we see another glimpse into this magical world of the infinite. It too will require a shift in our intuition.

To analyze this example we must first visit another famous infinite series:

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

85. Write out the next ten terms in this infinite series.
86. What is the sum of the first five terms in this series?
87. What is the sum the first ten terms in this series?
88. Show that your answers to the previous problems are close to $\frac{\pi^2}{6}$.

Showing, in 1734, that the sum of the infinite series $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ is precisely equal to $\frac{\pi^2}{6}$ was one of the great early triumphs of **Leonard Euler** (; -), one of the three greatest mathematicians of all time. It is also interesting to note that over 200 years later the sums of many closely related infinite series remain unknown. In fact, we do not know the sums of any of the *p-series* with odd exponents greater than one:

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots$$
$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \ldots$$
$$\frac{1}{1^7} + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} + \ldots$$
$$\vdots$$

Indeed, these sums are values of the *Riemann Zeta Function* which is the centerpiece of the *Riemann Hypothesis*. This unsolved problem is so important there is a $1,000,000 (US) prize for its solution as part of the Clay Mathematics Institute ***Millenium Prize Problems*** and it is the focus of many new books for general audiences. Links to this problem are considered in both of the volumes Discovering the Art of Mathematics: Number Theory and Discovering the Art of Mathematics: Patterns in this series.

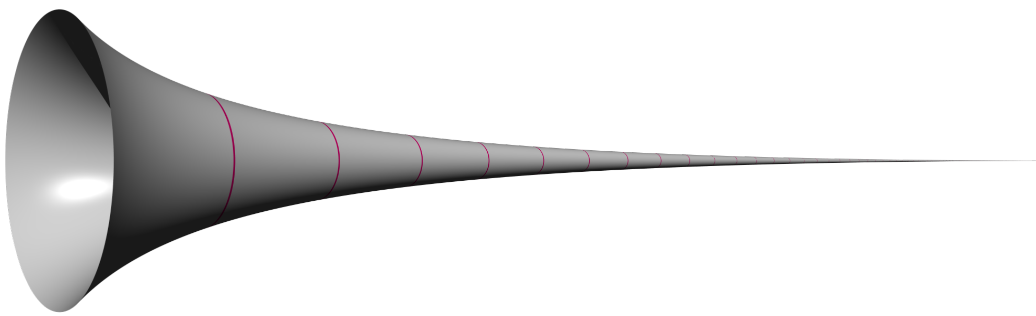

FIGURE 3. Gabriel's Horn.

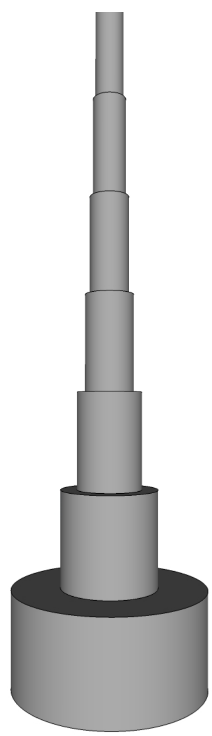

FIGURE 4. Gabriel's Wedding Cake.

The object in Figure 3 is called ***Gabriel's Horn***. It was invented by **Evangelista Toricelli** (Italian Mathematician and Physicist; -) in 1641 and had important links to the forthcoming invention of the calculus and sparked significant philosophical debates.[3] The object in Figure 4

[3]See "Gabriel's Wedding Cake" by Julian F. Fleron, *The College Mathematics Journal*, Vol. 30, No. 1, January 1999, pp. 35 - 38 for more details and references.

shares similarly surprising properties as Gabriel's Horn. It was invented by **Julian F. Fleron** (American Mathematician and Teacher; 1966 -) in 1998 and is called Gabriel's Wedding Cake.

Gabriel's Wedding Cake is constructed by piling an infinite number of cylindrical cake layers on top of one another. The height of each of these layers is one while the radii of these layers are $1, \frac{1}{2}, \frac{1}{3}, \ldots$, respectively.

89. Classroom Discussion: Can the volume and surface area of Gabriel's Horn and Wedding Cake be measured? Do you have predictions about the size of these measures?

90. Describe the formulae for the volume and suface area of a cylinder.

Find an expression, leaving π as a symbolic value without converting it to a decimal value, for the volume of ...

91. ... the first layer of the cake.
92. ... the second layer of the cake.
93. ... the third layer of the cake.
94. Describe the pattern you see in the measures of the volumes in the previous investigations. Use it to determine expressions for the volumes of the next four layers of the cake.
95. Determine an infinite series that measures the volume of the entire cake.
96. Using a series that we have previously considered, find the exact volume of the cake.
97. Find the total exposed area of the tops of the cakes. (Hint: it's easy if you think about all of them together, otherwise it is hard.)

Find an expression, leaving π as a symbolic value without converting it to a decimal value, for the surface area of the side of...

98. ... the first layer of the cake.
99. ... the second layer of the cake.
100. ... the third layer of the cake.
101. Describe the pattern you see in the measures of the surface areas in the previous investigations. Use it to determine expressions for the surface areas of the next four layers of the cake.
102. Use the previous investigation to determine an infinite series that represents the volume of the entire cake.
103. Using a series that we have previously considered, find the exact surface area of the cake.

104. Classroom Discussion: These investigations *prove* that the Gabriel's Wedding Cake has a volume of just over 5 cubic units while it has an infinite surface area. Does this seem feasible? If we had a single cake pan to make this cake we could make the batter to fill it, but could we grease the surface of the pan (so the cake didn't stick)? This seems problematic, doesn't it? Are there ways out of this dilemma? What does this do to our understanding of the infinite?

5. Connections

5.1. Diversity. By now you probably noticed that the overwhelming majority of our historical links have been to Eurocentric mathematicians. This parallels history in many areas. How much this has to do with the existing historical record versus issues of power, class, and race can be debated. In our particular context, progress is being made in uncovering powerful work of non-Eurocentric mathematicians. For example, many of the important results on representations of functions by *power series* seem to have been developed by Indian mathematicians of the Kerala region centuries before their European counterparts. In fact, some suggest that these results may have been carried back to Europe by Jesuit missionaries. To learn more one can begin with chapter "Indian Mathematics: The Classical Period and After" in The Crest of the Peacock: Non-European Roots of Mathematics by George Cheverghese Joseph and then look online for more recent discoveries.

5.2. Philosophy and Religion. In the introduction to Gabriel's Horn and Gabriel's Wedding Cake we used the word "invented". Do you think this is an appropriate word or do you think "discovered" would have been better? Those who believe "discovered" is a better word are generally called *Platonists*. Those who believe the more appropriate word is "invented" cannot be classified as easily, they may be *empiricists*, *constructivists*, *formalists*, etc. Indeed, the connections between mathematics and philosophy are deep and long-standing. Those with interest in pursuing more are encouraged to consult the chapter "From Certainty to Fallibility" in The Mathematical Experience by Philip J. Davis and Reuben Hersh which won the 1983 National Book Award in the Science category.

In the new book Naming Infinity Loren Graham and Jean-Michel Kantor describe how

> Name Worshipping - a religious viewpoint regarded as heresy by the Russian Orthodox Church and condemned by the Communist Party as a reactionary cult - influenced the emergence of a new movement in modern mathematics.

This movement was the continuation of the work of Georg Cantor on the infinite. The authors argue that "while the French were constrained by their rationalism, the Russians were energized by their mystical faith." The "Russian trio" of **Nikolai Nikolaevich Luzin** (Russian Mathematician; 1883 - 1950), **Dimitri Egorov** (Russian Mathematician; 1869 - 1931), and **Pavel Florensky** (Russian Mathematician; 1882 - 1937) had a deep impact on the development of a powerful Russian mathematical community in the twentieth century. They were clearly on the side of "invention" with Luzin speaking strongly about the importance of naming:

> Each definition is a piece of secret ripped from Nature by the human spirit. I insist on this: any complicated thing, being illumined by definitions, being laid out in them, being broken up into pieces, will be separated into pieces completely transparent even to a child, excluding foggy and dark parts that our intuition whispers to us while acting, separating into logical pieces, then only can we mover further, towards new successes due to definitions.

6. Further Investigations

6.1. Rearranging Paradox. Another way to see that rearranging infinite series effects the sum is to analzyze the alternating harmonic series as we did the the infinitely repeating decimal $0.999\ldots$ previously. Namely, define S by:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

F1. Multiply both sides of the equation for S by 2 to find an infinite series for $2S$.
F2. Combine terms which share the same denominators and show that the result is $2S = S$.
F3. We know $S \neq 0$, so divide by S to "show" $2 = 1$.
F4. Where is it that we have gone astray, or have we really proven that $2 = 1$?

6.2. Riemann's Rearrangement Theorem. Peter Lejeune Dirichlet (; -) is the earliest person we have on record of being aware that rearranging an infinite series may change its sum. He discovered this in 1827. It was not until 1852 that the question was investigated more fully, this time by Riemann who had sought the advice of his elder Dirichlet. The rearrangement theorem that now bears his name was proven by 1853 but was published posthumously in 1866, and only then because its importance was recognized by **Richard Dedekind** (German mathematician; 1831 - 1916).[4]

Riemann's Rearrangment Theorem is counterintuitive, showing that infinite series do not satisfy the *commutative law of addition* that we are used to. But in some sense it is maximally counterintuitive, showing that a conditionally convergent series can be rearranged to converge to *any* value one desires! With such surprises, one might expect that proving this result would be very difficult.

[4]See "Riemann's Rearrangement Theorem" by Stewart Galanor, *Mathematics Teacher*, vol. 80, no. 8, Nov. 1987, pp. 675-81 and A Radical Approach to Real Analysis by David Bressoud.

However, nothing could be further from the case. Rather, the direct and simple proof adds greatly to the beauty of this result.

Here we illustrate intuitively how the proof proceeds. A formal proof relies on little more than rigorous use of the key definitions of the ideas investigated here and below. Our approach follows that of Dunham in The Calculus Gallery who follows Riemann's original proof.

F5. Suppose the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent. Explain why the individual terms a_n must *converge* to 0 in the limit as $n \to \infty$.

F6. Explain how we can rearrange the infinite series $\sum_{n=1}^{\infty} a_n$, which may include both positive and negative terms, into the difference of two infinite series with all positive terms. I.e. $\sum_{n=1}^{\infty} a_n = (c_1 + c_2 + c_3 + \ldots) - (d_1 + d_2 + d_3 + \ldots)$ where $0 \leq c_n, d_n$. (Note: We chose the labels c and d to symbolize the "credits" and "debits" of the original series.)

F7. Suppose the infinite series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Explain why the associated infinite series of credits and debits (i.e. the series $c_1 + c_2 + c_3 + \ldots$ and $d_1 + d_2 + d_3 + \ldots$ from Further Investigation **F6**) both must diverge.

F8. Pick *any* target value, which we denote by T, you would like the rearranged series to converge to.

F9. Beginning with c_1 begin adding successive credits. Explain why the sum must eventually surpass T.

F10. If you stop adding successive credits as soon as the sum surpasses T, how far from T can the sum be?

F11. To the credits in Further Investigation **F10** begin subtracting successive debits starting at d_1. Explain why the sum must eventually fall below T.

F12. If you stop subtracting successive debits as soon as the sum falls below T, how far from T can the sum be?

F13. Explain how you can continue this process to rearrange the series so it has the form

$$(c_1 + c_2 + \ldots c_{n_1}) - (d_1 + d_2 + \ldots d_{m_1}) + (c_{n_1+1} + c_{n_1+2} + \ldots c_{n_2}) - (d_{m_1+1} + d_{m_1+2} + \ldots d_{m_2}) + \ldots$$

with partial sums mimicking those in Investigation **10** and Investigation **12**.

F14. Explain why the rearrangement just created must have sum T.

Most infinite series are remarkably hard to analyze, as the quote above from Abel seems to suggest. Whether their sum is finite or not can often be ascertained, but rarely can the exact value of the sum be found. Nonetheless, many important and remarkable infinite series have been discovered, several of them in the search for methods of approximating the ubiquitous constant .

The following infinite series is known as the **Gregory series**:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

F15. Write out the next ten terms in this series.

F16. Sum the first five terms in this series.

F17. Sum the first ten terms in this series.

F18. Show that your answers to the previous problems are close to /4.

Although it is hard to demonstrate, the Gregory series does has $\frac{\pi}{4}$ as a sum. That is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots,$$

a fact that was known not only by **Gregory** (; -), but **Leibniz** (; -), **Isaac Newton** (; -), and, almost two centuries earlier than any of these well-known mathematicians, by the Indian mathematician **Nilakantha** (; -).