Discovering the Art of Mathematics

Algebra (Draft)

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CHAPTER 1

Introduction

1. Creating an Algebra book

Blog post by Christine von Renesse, Sep. 2013:

In 2012 the Discovering the Art of Mathematics team started reaching out to the two-year colleges, wondering if our materials and pedagogy might be interesting and helpful to them. To start collaborating we went to the NEMATYC conference in 2013 and gave a workshop about some of our materials and ideas at `www.artofmathematics.org`. Among the many things we learned was the realization that many faculty are interested in support around algebra related courses.

So here was the question: could we use our materials (Discovering the Art for Mathematics books) to teach algebra related content? I went to the topic index on our website (`www.artofmathematics.org/books/topics`) and looked at all the chapters that are in the section algebra. The good news was that there is a lot of great material covering some basic understanding of algebra as well as some interesting practice and applications of algebra on our website. Unfortunately it is tedious (and not so pretty) to download the pdfs of the chapters and make them into a new book by merging them (I tried it). So I decided to actually grab our latex chapters and make it look a bit prettier. We dream of a system that will allow all users access to our latex chapters to create their own book versions, but realizing this vision will be some time off in the future.

The new algebra book has now a table of content that follows a flow from easier to more advanced or applied algebra topics. Please let us know if this is useful for your class and if there are particular topics that you are missing.

And while I was writing this blog post, I received an email from a Community College in Illinois where they started teaching a new course: Preparatory Math for General Education (PMGE). This course will replace the traditional beginner and intermediate algebra courses and teach algebra topics conceptually and problem based - a pathway to a mathematics for liberal arts course. This sounds like a great place to use some of our ideas. I would love to know if we would need to adjust the level of our activities or not, my guess is that we would. Phil Hotchkiss and Volker Ecke will travel to Springfield, IL in October 2013 to do professional development at a community college and I hope they will learn more about how to adapt our activities for the needs of a PMGE class.

2. How to use this book in a classroom

The selected chapters will help students gain a deeper conceptual understanding of some concepts in algebra. The focus is on connetions with the liberal arts, understanding **why** rules in algebra work and relating algebra to patterns and other areas in mathematics. There are no practice questions provided, since we concentrate on creating questions that foster deep thinking about mathematics.

These chapters will not be enough to "cover the material" of a typical algebra class. We hope, however, that you can use a few chapters to supplement your typical algebra course and make it more *alive* for your students. We are currently writing a pedagogy-guide about our version of an inquiry-based classroom; for now please contact us if you have any questions about how exactly we use the materials and how we run our classes.

3. Connections to other Books

The chapters in the algebra book are taken from the following books:

1. "Number Proofs" is a chapter in the Reasoning Book.
2. "Linear Patterns and Functions" is a chapter in the Pattern Book.
3. "Quadratic Growth and Probelms Solving with Patterns" is a chapter in the Pattern Book.
4. "The Use of Patterns and Language in the Creation of Powerful Number Systems" is a chapter in the Pattern Book.
5. "Existence of $\sqrt{-1}$" is a chapter in the Reasoning Book.
6. "Applications of Algebra: Tuning and Intervals" is a chapter in the Music Book.

There are also a few chapters in the Number Theory book that would be great applications of algebra.

Contents

Preface: Notes to the Explorer

Yes, that's you - you're the explorer.

"Explorer?"

Yes, explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a traditional book. Indeed, this book cannot be read in the traditional sense. For this book is really a guide. It is a map. It is a route of trail markers along a path through part of the world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore this path - to take a surprising, exciting, and beautiful journey along a meandering path through a mathematical continent named the infinite. And this is a vast continent, not just one fixed, singular locale.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by dozens of exercises that closely mimic illustrative examples. Rather, after a brief introduction to the chapter, the majority of each chapter is made up of Investigations. These investigations are interwoven with brief surveys, narratives, or introductions for context. But the Investigations form the heart of this book, your journey. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover the infinite. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking, mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is discovered each day than in any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! *Fermat's Last Theorem*, which is considered in detail in Discovering that Art of Mathematics - Number Theory, was solved in 1993 after 350 years of intense struggle. The 1\$ Million Poincaŕe conjecture, unanswered for over 100 years, was solved by **Grigori Perleman** (Russian mathematician; 1966 -). In the time period between when these words were written and when you read them it is quite likely that important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning mathematical skills, mathematical reasoning, and mathematical applications. Arithmetical and statistical skills are useful skills everybody should possess. Who could argue with learning to reason? And we are all aware, to some degree or another, how mathematics shapes our technological society. But there is something more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is one of its driving forces. As the famous **Henri Poincaŕe** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies
> it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as both a liberal art and as a science. As a powerful science, mathematics shapes our technological society and serves as an indispensable tool and language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in many other fine, accessible books (e.g. [COM] and [TaAr]). Instead, our purpose here is to journey down a path that values mathematics from its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was rescued and preserved in monasteries. Knowledge was categorized into the classical liberal arts and mathematics made up several of the seven categories.[1] During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts - except in the contemporary classrooms and textbooks where the focus of attention has shifted solely to the training of qualified mathematical scientists. If you are a student of the liberal arts or if you simply want to study mathematics for its own sake, you should feel more at home on this exploration than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more. In your exploration here you will see that mathematics is a human endeavor with its own rich history of human struggle and accomplishment. You will see many of the other arts in non-trivial roles: art and music to name two. There is also a fair share of philosophy and history. Students in the humanities and social sciences, you should feel at home here too.

Mathematics is broad, dynamic, and connected to every area of study in one way or another. There are places in mathematics for those in all areas of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty
> cold and austere, like that of sculpture, without appeal to any part of our weaker
> nature, without the gorgeous trappings of paintings or music, yet sublimely pure and
> capable of a stern perfection such as only the greatest art can show.

It is my hope that your discoveries and explorations along this path through the infinite will help you glimpse some of this beauty. And I hope they will help you appreciate Russell's claim that:

> ...The true spirit of delight, the exaltation, the sense of being more than [hu]man,
> which is the touchstone of the highest excellence, is to be found in mathematics as
> surely as in poetry.

Finally, it is my hope that these discoveries and explorations enable you to make mathematics a real part of your lifelong educational journey. For, in Russell's words once again:

> ...What is best in mathematics deserves not merely to be learned as a task but to be
> assimilated as a part of daily thought, and brought again and again before the mind
> with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have served as beacons to people who have explored the continents of mathematics throughout history.

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[1]These were divided into two components: the *quadrivium* (arithmetic, music, geometry, and astronomy) and the *trivium* (grammar, logic, and rhetoric); which were united into all of knowledge by philosophy.

Navigating This Book

Before you begin, it will be helpful for us to briefly describe the set-up and conventions that are used throughout this book.

As noted in the Preface, the fundamental part of this book is the Investigations. They are the sequence of problems that will help guide you on your active exploration of mathematics. In each chapter the investigations are numbered sequentially. You may work on these investigation cooperatively in groups, they may often be part of homework, selected investigations may be solved by your teacher for the purposes of illustration, or any of these and other combinations depending on how your teacher decides to structure your learning experiences.

If you are stuck on an investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us: "If thee is no struggle, there is no progress." Keep thinking about it, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Investigation numbers are bolded to help you identify the relationship between them.

Independent investigations are so-called to point out that the task is more significant than the typical investigations. They may require more involved mathematical investigation, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The Connections sections are meant to provide illustrations of the important connections between mathematics and other fields - especially the liberal arts. Whether you complete a few of the connections of your choice, all of the connections in each section, or are asked to find your own connections is up to your teacher. But we hope that these connections will help you see how rich mathematics' connections are to the liberal arts, the fine arts, culture, and the human experience.

Further investigations, when included are meant to continue the investigations of the area in question to a higher level. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory.

Within each book in this series the chapters are chosen sequentially so there is a dominant theme and direction to the book. However, it is often the case that chapters can be used independently of one another - both within a given book and among books in the series. So you may find your teacher choosing chapters from a number of different books - and even including "chapters" of their own that they have created to craft a coherent course for you. More information on chapter dependence within single books is available online.

Certain conventions are quite important to note. Because of the central role of proof in mathematics, definitions are essential. But different contexts suggest different degrees of formality. In our text we use the following conventions regarding definitions:

- An *undefined term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An *informal definition* is italicized and bold faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.
- A **formal definition** is bolded the first time it is used. This is a formal definition that suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a biographical name appears it is bolded and basic biographical information is included parenthetically to provide some historical, cultural, and human connections.

Number Proofs

Proof is an idol before which the mathematician tortures himself.

Sir Arthur Eddington (; -)

A elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline (; -)

A good proof is one that makes us wiser.

Yu. I. Manin (; -)

1. Odd and Even Numbers

1. Explain what even and odd numbers are.

2. We often explain things intuitively. If you had to give a rigorous definition of even numbers, what would it be? What about odd numbers? Explain.

The typical definition of an **even number** is:

A positive integer is even if it can be written as $2n$ where n is some non-negative integer.

The definition of an **odd number** is analogous:

A positive integer is odd if it can be written as $2n + 1$ where n is some non-negative integer.

3. Are your definitions in 2 equivalent to those just given? If so, prove your result. If not, provide an example which illustrates the difference.

4. Take several pairs of odd counting numbers and multiply each pair together. What do you notice about the products of these pairs of odd counting numbers?

5. Using the pattern you have observed in 4, state a conjecture that characterizes the product of any two odd counting numbers.

Here we demonstrate how this result can be proven deductively:

PROOF. Denote the two counting numbers by a and b. By assumption, both a and b are odd. By definition this means that there are positive integers n and m so that $a = 2n + 1$ and $b = 2m + 1$. Then the product $a \cdot b$ is given by:

(1)
$$a \cdot b = (2n + 1) \cdot (2m + 1)$$

(by the distributive law)
$$= 4nm + 2n + 2m + 1$$

(2)
$$= 2\,(2nm + n + m)) + 1$$

$2nm + n + m$ is a positive integer and so, by definition, $a \cdot b$ is odd. □

6. Take several pairs of even counting numbers and add each pair together. What do you notice about the sums of these pairs of even counting numbers?

7. Using the pattern you have observed in 10 state a conjecture that characterizes the sum of any two even counting numbers.

8. Using the definition of even numbers, prove your conjecture about the sum of two even numbers.

9. Do you see a way to prove your conjecture using your definition in 2? Explain.

10. Take several pairs of odd counting numbers and add each pair together. What do you notice about the sums of these pairs of odd counting numbers?

11. Using the pattern you have observed in 10 state a conjecture that characterizes the sum of any two odd counting numbers.

12. Prove your conjecture about the sum of two odd numbers.

13. Take several pairs of counting numbers, one even and one odd, and multiply each pair together. What do you notice about the products of these pairs, one even and one odd, of counting numbers?

14. Using the pattern you have observed in 13 state a conjecture that characterizes the product of any two, one even and one odd, counting numbers.

15. Prove your conjecture about the product of an even and an odd number.

2. Proofs Without Words: Gauss Sum

Mathematical folklore holds that the great **Carl Freidrich Gauss** (; -) was once, as a very young child, scolded by being sent to the coat closet with a slate to determine the sum of the first hundred numbers: $1 + 2 + 3 + \ldots + 99 + 100$. The legend holds that he returned within a minute with the correct answer.

Figure 1 illustrates Gauss's method as it can be represented with blocks to determine the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.
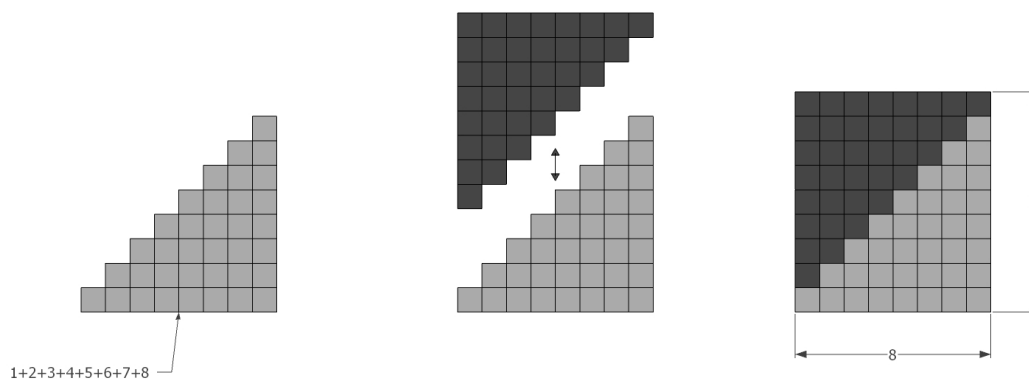
FIGURE 1. Determining the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.

16. Use Gauss's method to determine the sum Gauss was required to compute.

17. Use Gauss's method to determine the sum $1 + 2 + 3 + \ldots + 1,000,000,000,000$.

18. Suppose that n is a positive integer. Find an algebraic expression for the value of the sum $1 + 2 + 3 + \ldots (n-2) + (n-1) + n$.

19. Check that your result agrees with your answers to the two explicit problems computed previously.

20. Explain how Figure 2 provides a *proof without words* which proves the general result in 41

21. Determine the value of the following sums:
- $1 + 2 + 1$
- $1 + 2 + 3 + 2 + 1$
- $1 + 2 + 3 + 4 + 3 + 2 + 1$
- $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$

22. What pattern do you see? Describe this pattern using the language of an algebraic equation.

1+2+3+4+5

(n-4) + (n-3) + (n-2) + (n-1) + n

n+1

n
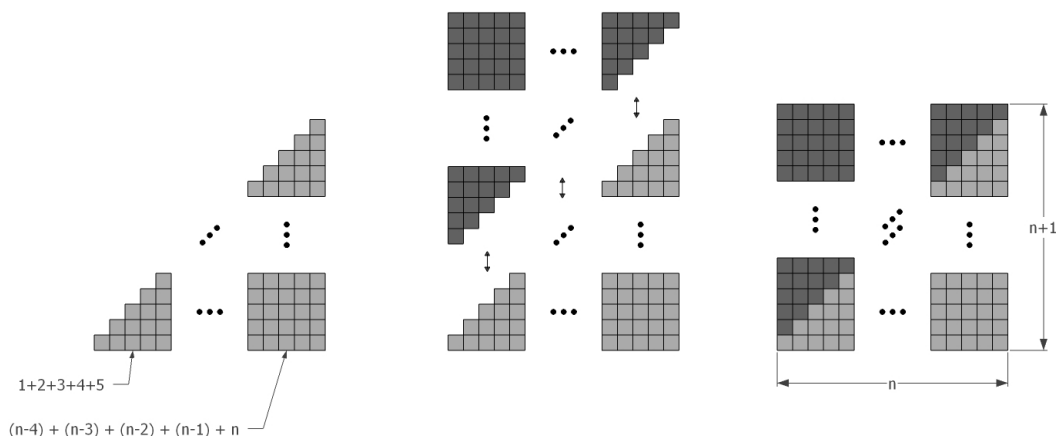
FIGURE 2. Determining the sum $1 + 2 + 3 + \ldots (n-2) + (n-1) + n$.

23. Find a proof without words for this result.

24. Determine the value of the following sums:
- $1 + 3$
- $1 + 3 + 5$
- $1 + 3 + 5 + 7$
- $1 + 3 + 5 + 7 + 9$

25. What pattern do you see? Describe this pattern using the language of an algebraic equation.

26. Find a proof without words for this result.

3. Mathemagical Tricks

27. Choose two single-digit numbers. Then perform the following computations in order:
- Multiply the first number by 2
- Add 3 to the result
- Multiply the sum by 5
- Add the second number to the product
- Multiply the sum by 10.

Show these computations step by step and write down the end result.

A mathemagician - human in the form of your teacher or a peer, or online at ???, will now divine the identity of your two numbers from the result of your computation.

28. Do you think this is a compelling trick? Explain.

29. In an effort to understand how this trick worked, compile a list of beginning numbers and the final computations.

30. From this list in Investigation **29**, can you determine how it was that the mathemagician divined the two numbers in question? Explain.

31. It wouldn't be much of a trick if it only worked sometimes. Use algebra to prove that this trick will work for any pair of beginning numbers.

Here's another trick.

32. Choose a secret number. Then perform the following computations:
- Add 1 to the number chosen
- Multiply the sum by 3

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- Add the square of the original number to this product
- Multiply the sum by 4
- Subtract 3 from the product
- Take the square root of the difference.

Show these computations step by step and write down the end result.

A mathemagician - human in the form of your teacher or a peer, or online at ???, will now divine the identity of your two numbers from the result of your computation.

33. Do you think this is a compelling trick? Explain.

34. In an effort to understand how this trick worked, compile a list of beginning numbers and the final computations.

35. From this list in Investigation **34**, can you determine how it was that the mathemagician divined the two numbers in question? Explain.

36. It wouldn't be much of a trick if it only worked sometimes. Use algebra to prove that this trick will work for any beginning number.

Linear Patterns and Functions

In 1953 I realized that the straight line leads to the downfall of mankind. But the straight line has become an absolute tyranny. The straight line is something cowardly drawn with a rule, without thought or feeling; it is the line which does not exist in nature... Any design undertaken with the straight line will be stillborn. Today we are witnessing the triumph of rationalist knowhow and yet, at the same time, we find ourselves confronted with emptiness. An esthetic void, desert of uniformity, criminal sterility, loss of creative power. Even creativity is prefabricated. We have become impotent. We are no longer able to create. That is our real illiteracy.

Friedensreich Regentag Dunkelbunt Hundertwasser (Austrian Artist and Architect; 1928 - 2000)

The whole science of geometry may be said to owe its being to the exorbitant interest which the human mind takes in lines. We cut up space in every direction in order to manufacture them.

William James (American Psychologist; 1842 - 1910)

1. Introduction: The Line

Certainly there are significant limitations to a world populated only by the lowly line. Art would certainly be relatively crippled if it could employ only lines, limiting us to *line art* and *string art* like those pieces shown in Figures 1, 2, and 13. Perhaps that is what Hundertwasser meant in his lengthy quote above, or when he called the straight line "ungodly". Sections of Discovering the Art of Geometry in this series show us how *fractals* are one way mathematics has freed itself from it's most basic objects: lines, circles, and spheres.
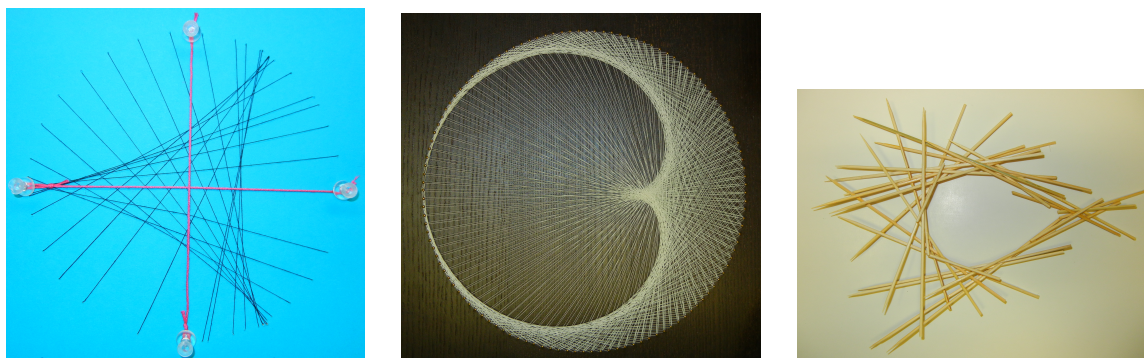
FIGURE 1. Original student string art.

Appreciating the freedoms that we can find in mathematics as well as art when we are freed from lines, we think Hundertwasser's condemnation is a bit too harsh. We believe there is significant merit in

William James' view - the line has natural resonances with the human mind. Throughout this chapter we provide short asides that illustrate the power and beauty of mathematical objects that are *linear* or *arithmetic*. We hope this removes some of the line's stigma.

FIGURE 2. Line Art

There is also a practical side to our approach. Students of mathematics as well as of art are often well-served by starting with a limited slate of objects to work with as they begin to explore these subjects. Once they have some familiarity with the general principles, ideas, and methods of these arts then the palets can be expanded fruitfully.

2. Chronophotography

We get a *sequence* of *equally timed* images. If we run through these images in *linear* time we get - a movie, television images, cartoons and animation, video! A basic way to see this is using flip books.

The most important early work on chronophotgraphy was done by Marey and Muybridge. These "time and motion" studies are still fundamental to artists and animators - as well as many connections to other areas. Human locomotion was one of the first areas studied. To determine the movements of a human walking Marey dressed a man in a black velvet suit and had reflective *lines* along his upper spine, arm, and leg, as shown on the left in Figure **??**. The result is the striking image on the right in Figure **??**. This image and similar ones due to Maybridge were the impetus for Marcel Duchamp's Nude Descending Staircase (1912), one of the more important works of the early *Modernist* movement in art. The relationship is clear, as is the fundamental role that lines play.
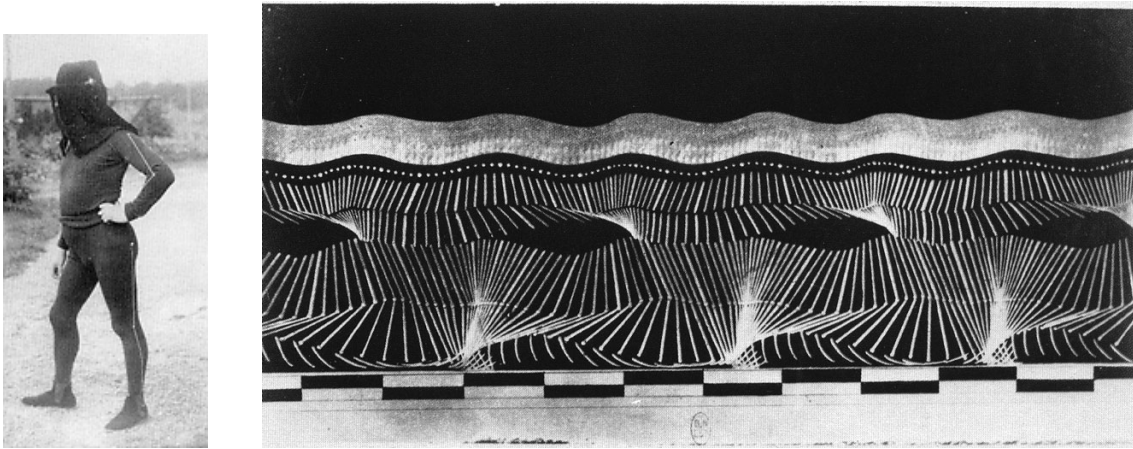
FIGURE 3. Man in black velvet and **tienne-Jules Marey** (French Scientist and Photographer; 1830 - 1904)
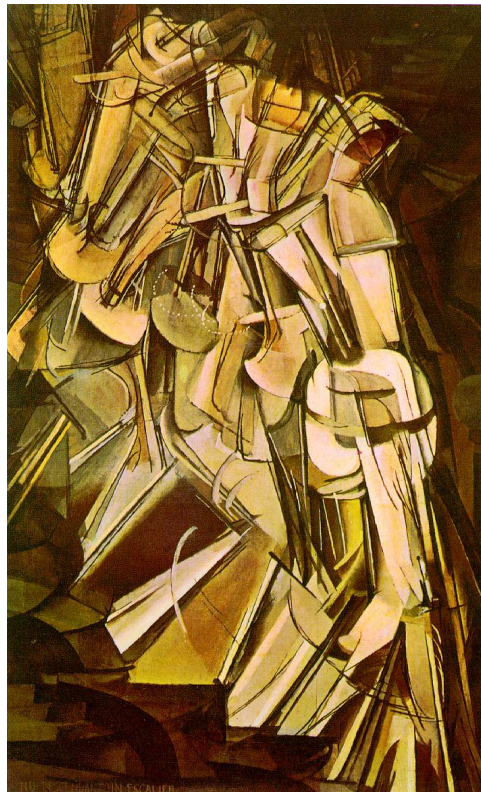
FIGURE 4. Nude Descending Staircase by **Marcel Duchamp** (French Artist; 1887 - 1968)

3. A First Linear Function - The Darbi-i Imam Frieze
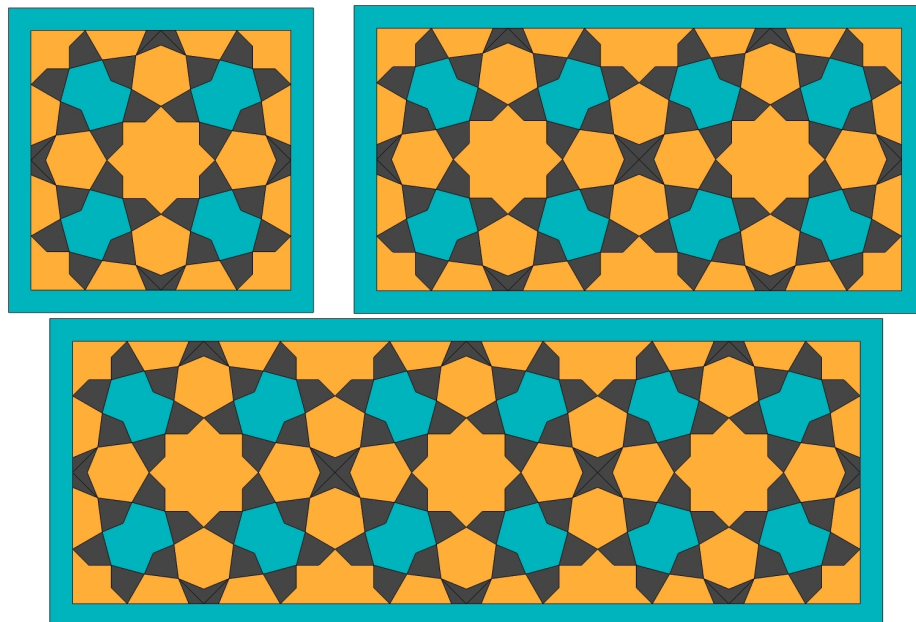
FIGURE 5. First three stages from a model of the *frieze pattern* on the Darbi-i Imam shrine.

In Section **??** we talked about different ways to represent data. Figure 5 are models which represent the first three stages in the construction of a tile *frieze* (a pattern which extends periodically in one direction). Made out of ceramic tiles like those which are found in mosaics throughout the world, the particular frieze being modeled is the entry portal of the Darbi-i Imam Shrine in Isfahan, Iran. The original frieze is pictured in Figure 9. These models provide **physical representation** of an underlying pattern.

1. If the frieze pattern in Figure 5 keeps growing in the *evident*[1] way, draw major features of the next three stages in this pattern.

Suppose that you were building this tilework frieze. It is of interest to know how many tiles of each type will be needed.

One **numerical representation** for the number of turquoise octagons required to complete the different stages of this pattern is the **sequence**:

$$4, 8, 12, \ldots$$

Each entry in a sequence is called a **term** in the sequence.

2. Assuming the frieze pattern continues in the evident way, what are the next four terms in the turquoise octagon sequence?

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[1]As noted in the Student Toolbox, any finite number of terms create infinitely many patterns. E.g. given the four terms $1, 3, 5, 7$, this pattern can be extended as $1, 3, 5, 7, 9, 11, 13, \ldots$ as odds, or as $1, 3, 5, 7, 11, 13, 17, \ldots$ as primes, or as $1, 3, 5, 7, 1, 3, 5, 7, 1, 3, 5, 7, \ldots$ just because, or ... So, when you see the word *evident* way it is just a caveat that we're hoping you might see the pattern the way we intended and not some unique way of your own. We'll continue to emphasize the term as a reminder.

The terms in a sequence come in a specific order and we specify their location using **subscripts**; if we use the letter t to represent the turquoise sequence then t_1 represents the first term, t_2 the second term, etc. I.e. $t_1 = 4, t_2 = 8, t_3 = 12, \ldots$

3. What is the tenth term in the turquoise octagon sequence? I.e. $t_{10} = ?$

4. What is the fiftieth term in the turquoise octagon sequence? I.e. $t_{50} = ?$

Another numerical representation for the number of turquoise octagons that make up this pattern is as a **table of values**:

n	t
1	4
2	8
3	12
⋮	⋮

5. Fill in the indicated values in the table for the turquoise octagon:

n	t
1	4
2	8
3	12
4	☐
5	
⋮	⋮
25	☐
⋮	⋮
75	☐
⋮	⋮
1000	☐
⋮	⋮
n	☐

6. By filling in the line representing the number of tiles needed in the nth stage you have found an **algebraic representation** for the pattern. Write this representation as an algebraic formula:
$$t_n = \boxed{}.$$

One way to look for patterns in numerical data is to compare successive terms in a sequence or table by considering their **first differences** as illustrated in Figure Figure 3. We use the capital Greek letter Δ to represent the first differences.

The growth of a numerical pattern is called **arithmetic** whenever the first differences in a sequence or table of values is constant.

Graphical representations are also important. There are many ways to represent our turquoise octagon data as a graph, as illustrated in Figure **??**.

Notice that the variables t and n continue to represent the number of turquoise octagons and the stage number respectively. The standard convention is to have the *independent variable* on the horizontal axis and the *dependent variable* on the vertical axis. Typically the Cartesian graph more regularly than the bar graph.

7. Extend the graph of the turquoise data to include all values of $n \leq 8$.

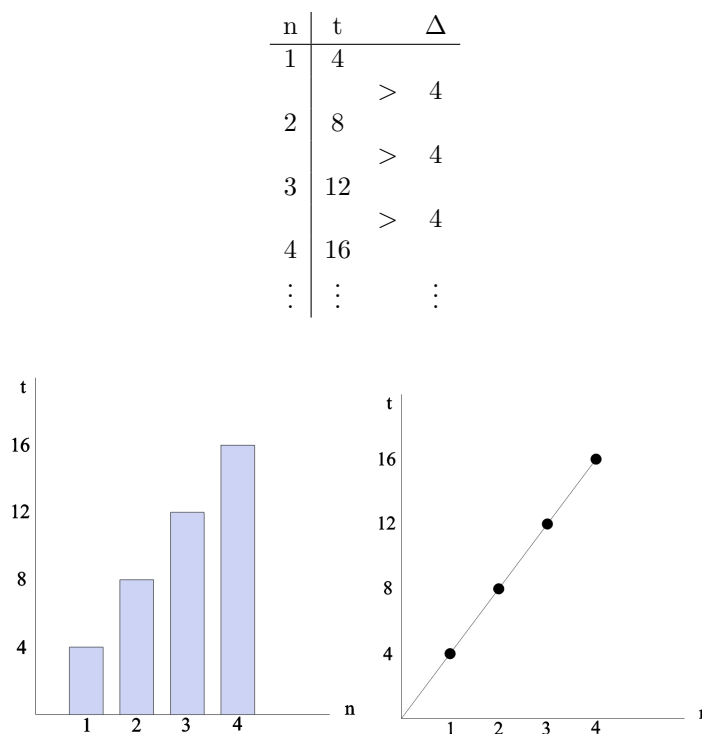| n	t	Δ
1	4	
		> 4
2	8	
		> 4
3	12	
		> 4
4	16	
⋮	⋮	⋮

FIGURE 6. **Cartesian graph** and **Bar graph** of the turquoise octagon pattern.

8. The graph of the turquoise data in Figure 6 is called **linear**. Explain why, providing a working definition of the term linear for use hereafter.

9. You should already know about the *slope* of a linear graph. Give a working definition of what the slope of a linear graph is. Then determine the slope of the graph in Figure 6.

10. You should know about the *vertical intercept* of a graph.[2] Give a working definition of what the vertical intercept of a graph is. Then determine the vertical intercept of the graph in Figure 6.

Functions of the form $f = m \cdot n + b$ where m and b are fixed numbers are called **linear functions**.

Now consider the beige hexagon tiles in the Darbi-i Imam frieze. Let's use the dependent variable b to represent the number of beige hexagons at each stage, counting only the number of whole hexagons that appear.

11. Represent the pattern of beige hexagons numerically as both a sequence and as a table of values. Provide six or eight terms of each.

12. Compute the first differences of both of the numerical representations in 11. Describe these first differences.

13. Is the pattern of beige hexagons in the frieze arithmetic?

14. Represent the pattern of beige hexagons in the frieze pattern graphically.

15. Is the graph in 14 linear? If not, how can it be described? If so, what are its slope and vertical intercept?

16. Represent the pattern of beige hexagons in the frieze pattern algebraically.

17. Is the function that describes the pattern of beige hexagons in the frieze pattern linear?

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[2]When we denote the vertical axis by the dependent variable y the intercept is generally known as the y-intercept.

4. Paradigm Shift - The Darbi-i Imam Tessellation

In the study of **crystals** (e.g. diamonds, salt, ice, snowflakes, and quartz), five-fold symmetry was not seen and it was long thought that such symmetry could not exist in a natural crystal. This occurs because crystals that are created by nicely ordered structures all appear to have **translational symmetry**, that is, they repeat periodically in a natural way.

Analogously, it was thought that in two-dimensional tilings, any collection of tiles that could *tessellate* could also be made to tessellate in a periodic way.

Mathematicians, physical scientists, artists, and craftspeople have thought about crystals and tilings throughout human history. So it was a paradigm shift when in the 1960's it became clear that **aperiodic tiles**, tiles that would tessellate but could *never* be made to tessellate periodically, existed. In 1973 **Roger Penrose** (English physicist and mathematician; -) discovered a remarkably simple set of two tiles that were aperiodic. Shown in Figure 7, are Penrose's **Kites and Darts** which can be put together in many different ways to tessellate the plane, but cannot tessellate the plane in a periodic way.
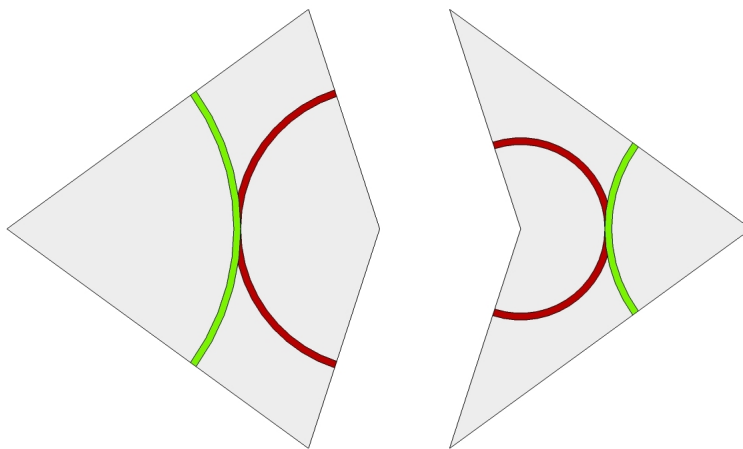

FIGURE 7. Penrose's kites and darts. (Note: Colors along edges must match for adjacent tiles.)

This celebrated discovery ushered in new urgency in the attempt to find three-dimensional analogues; naturally occurring **quasicrystals**. They were found in 1982 by **Dan Shechtman** (Israeli physicist; -). Unfortunately, they were so revolutionary to accepted scientific doctrine that Shechtman was, in his own words, "a subject of ridicule and lectured about the basics of crystallography. The leader of the opposition to my findings was the two-time Nobel Laureate Linus Pauling, the idol of the American Chemical Society and one of the most famous scientists in the world." Pauling went so far to say, in reference to Shechtman, "There is no such thing as quasicrystals, only quasi-scientists." Shechtman perservered, saying, "For years, 'til his last day, he [Pauling] fought against quasi-periodicity in crystals. He was wrong, and after a while, I enjoyed every moment of this scientific battle, knowing that he was wrong."[3]

Shechtman's perserverence paid off. On 5 October, 2011, he was awarded the Nobel Prize in Chemistry "for the discovery of quasi-crystals". The Press Release by the Nobel Committee is particularly interesting:

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[3]From "Ridiculed crystal work wins Nobel for Israeli" by Patrick Lannin and Veronica Ek, Reuters, 8/5/2011.

A remarkable mosaic of atoms

In quasicrystals, we find the fascinating mosaics of the Arabic world reproduced at the level of atoms: regular patterns that never repeat themselves. However, the configuration found in quasicrystals was considered impossible, and Dan Shechtman had to fight a fierce battle against established science. The Nobel Prize in Chemistry 2011 has fundamentally altered how chemists conceive of solid matter.

On the morning of 8 April 1982, an image counter to the laws of nature appeared in Dan Shechtman's electron microscope. In all solid matter, atoms were believed to be packed inside crystals in symmetrical patterns that were repeated periodically over and over again. For scientists, this repetition was required in order to obtain a crystal.

Shechtman's image, however, showed that the atoms in his crystal were packed in a pattern that could not be repeated. Such a pattern was considered just as impossible as creating a football using only six-cornered polygons, when a sphere needs both five- and six-cornered polygons. His discovery was extremely controversial. In the course of defending his findings, he was asked to leave his research group. However, his battle eventually forced scientists to reconsider their conception of the very nature of matter.

Aperiodic mosaics, such as those found in the medieval Islamic mosaics of the Alhambra Palace in Spain and the Darb-i Imam Shrine in Iran, have helped scientists understand what quasicrystals look like at the atomic level. In those mosaics, as in quasicrystals, the patterns are regular - they follow mathematical rules - but they never repeat themselves.

When scientists describe Shechtman's quasicrystals, they use a concept that comes from mathematics and art: the *golden ratio*. This number had already caught the interest of mathematicians in Ancient Greece, as it often appeared in geometry. In quasicrystals, for instance, the ratio of various distances between atoms is related to the golden mean.

Following Shechtman's discovery, scientists have produced other kinds of quasicrystals in the lab and discovered naturally occurring quasicrystals in mineral samples from a Russian river. A Swedish company has also found quasicrystals in a certain form of steel, where the crystals reinforce the material like armor. Scientists are currently experimenting with using quasicrystals in different products such as frying pans and diesel engines.

You may be surprised to see mention of the Darbi-i Imam shrine in this citation, aperiodic tilings and quasi-crystals as we have described them are recent discoveries.

Once again, as with the geocentric model of the solar systems and a non-flat earth, this is largely a function of our self-importance and our lack of respect for the brilliance of the ancient scholars, artists and craftspeople.

In 2005, while visiting the Middle East, graduate student **Peter Lu** (; -) became very interested in the tile-work on the Darbi-i Imam shrine. When he returned to Harvard he set to work studying these tilings. What he discovered was shocking:

The asymptotic ratio of hexagons to bowties approaches the golden ratio τ (the same ratio as kits to darts in a Penrose tiling), an irrational ratio that shows explicitly that the pattern is quasi-periodic. Moreover, the Darbi-i Imam tile pattern can be mapped directly into Penrose tiles.[4]

Iranian craftspeople had predated the discoveries of Penrose by over 500 years!

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[4]From "Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture" by Peter J. Lu and Paul J. Steinhardt, *Science*, Vol. 314, 23 February, 2007, pp. 1106-1110.

The (combined) mathematical and artistic study of medieval tilings such as these is rich and beautiful.[5]

FIGURE 8. Entry portal from Dabri-i Imam in Isfahan, Iran. Photo courtesy of Peter Lu.

FIGURE 9. Darbi-i Imam frieze detail.

18. What are some other examples in science and/or mathematics where there were particularly personal attacks/disagreements between scientists?

19. What would you have done had you found yourself in Shechtman's place in this controversy?

---

[5]One fun place to start is building with these tiles. See `http://www.3dvinci.net/mathforum/GirihTiles_StudentVersion.pdf` where Google SketchUp is used to help create tilings with the tiles that are found in the Darbi-i Imam.

Figure 10. Peter Lu.

5. Multiple Representations of Linear Functions

The frieze patterns above appeared as physical patterns. But we can just as well start with a function represented algebraically. For example,

$$f = 3 \cdot n + 7.$$

20. Is the function f linear?

21. Represent the function f numerically as both a sequence and as a table of values. Provide six or eight terms of each.

22. Compute the first differences of both of the numerical representations in Investigation **21**. Describe these first differences.

23. Is the numerical data formed by f an arithmetic pattern?

24. Represent the function f graphically.

25. Is the graph in 24 linear? If not, how can it be described? If so, what are its slope and vertical intercept?

26. What about physically? Suppose you worked with tiles. Is there a natural way to show how we can represent the function f physically as a growing pattern of tiles?

Now consider new data, which is given graphically as in Figure 11. Assume that this graph continues in the *evident* way.

27. Is the graph in 11 linear? If not, how can it be described? If so, what are its slope and vertical intercept?

28. Represent the function g numerically as both a sequence and as a table of values. Provide six or eight terms of each.

29. Compute the first differences of both of the numerical representations in 28. Describe these first differences.

30. Is the numerical data formed by g an arithmetic pattern?

31. Represent the function g algebraically.

32. Is the function g linear?

33. Represent the function g physically.

Consider the sequence s given by $7, 11, 15, 19, \ldots$
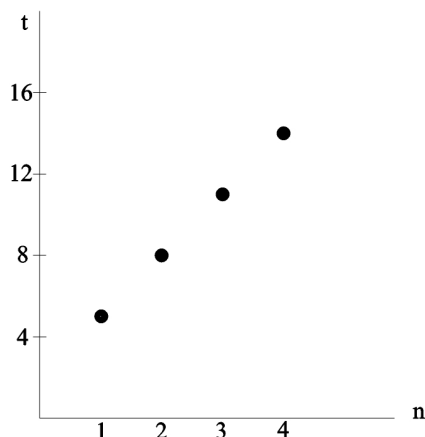
FIGURE 11. Graph of the data from a function g.

34. Represent the function s numerically as a table of values. Provide six or eight terms.
35. Compute the first differences of both of the numerical representations in 34. Describe these first differences.
36. Is the numerical data formed by s an arithmetic pattern?
37. Represent the function s graphically.
38. Is the graph in 37 linear? If not, how can it be described? If so, what are its slope and vertical intercept?
39. Can you find a way to represent the function s physically.
40. Represent the function s algebraically.
41. Is the function s linear?

Now it's time to make your own arithmetic pattern as a growing frieze. Figure 12 below shows a growing arithmetic pattern that is constructed using **pattern blocks**, a collection of six different, brightly colored tile blocks that are often used in elementary school mathematics classrooms.

42. Use pattern blocks or one of the online pattern block applets (e.g. `http://nlvm.usu.edu/en/nav/frames_asid_170_g_2_t_3.html`) to build a growing frieze pattern where at least one of the types of blocks illustrates arithmetic growth.
43. Graph your arithmetic pattern.
44. Find the algebraic representation of your pattern.

6. Meta Patterns: Linear Functions in General

We have been considering single functions/patterns to see if they were linear/arithmetic. Now we would like to see if there are patterns that unite what we have learned about these patterns. Such a pattern could be called a *meta pattern*.

45. If you have a linear function $f = m \cdot n + b$ what can you say about its graphical representation? Its numerical representations? Its physical representation? Explain.
46. If you have a function whose graph is linear, what can you say about its algebraic representation? Its numerical representation? Its physical representation? Explain
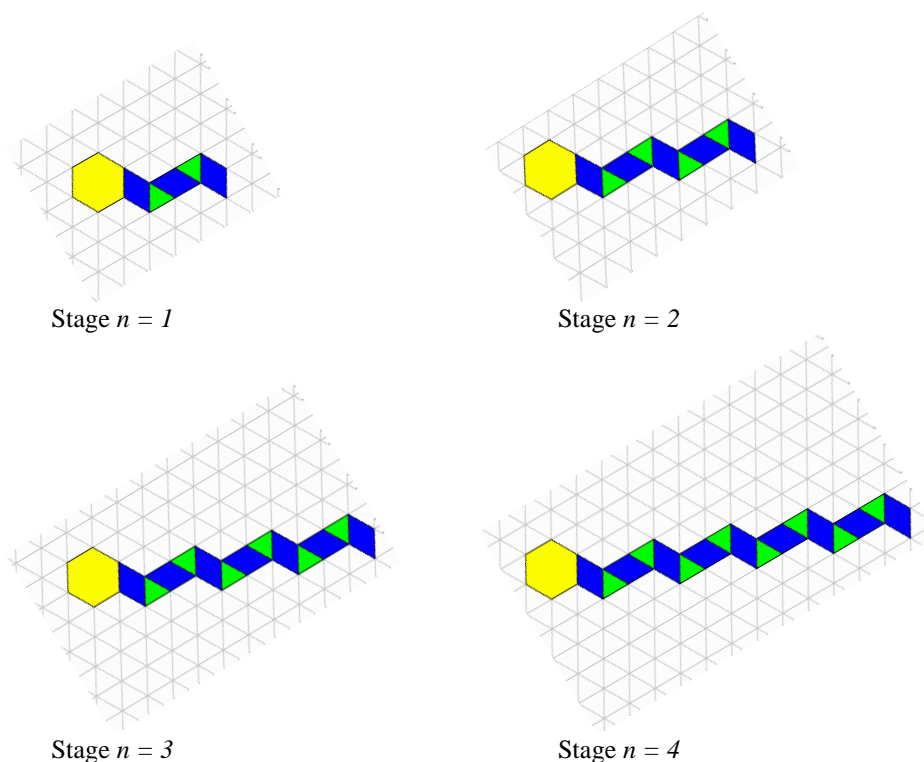
Stage *n = 1*   Stage *n = 2*

Stage *n = 3*   Stage *n = 4*

FIGURE 12. Growing caterpillar pattern constructed from pattern blocks.

47. If you have a function whose numerical data is arithmetic, what can you say about its algebraic representation? Its graphical representation? Its physical representation? Explain.

7. Connections

[Some of these maybe should be included above to break up the other sections.]

7.1. Pattern Block Patterns. It is difficult to overemphasize the power of simple manipulatives like pattern blocks to nurture the creative spirit. By all means, try to find the opportunity to create your own mosaic.

For those interested in teaching, there are many wonderful resources which describe or model the use of such manipulatives in elementary teaching. For examples, "Case 19: Growing Worms 1" and "Case 20: Growing Worms 2" in Discovering Mathematical Ideas: Patterns, Functions and Change Casebook by Deborah Schifter, Virginia Bastable and Susan Joe Russell.

7.2. Linear Programming.

The linear-programming was – and is – perhaps the single most important real-life problem.[6]

Keith Devlin (; -)

7.3. Origami. Origami EVERY one of these shapes is made by folding lines!!! There is nothing else.

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[6]From Mathematics: The New Golden Age, p. 605.

7.4. Shadows. Light/CAT scans/Shadows All of these things are just the lines made by light. CAT scans are just lines and look what they tell us about our 3D bodies!!

7.5. Linear Regression. Linear Regression A fundamental application. That it is applied so much gives us some sense of how many things exhibit approximately linear growth.

7.6. Art. Perspective Drawing Give links to the appropriate sections in the Geometry book.
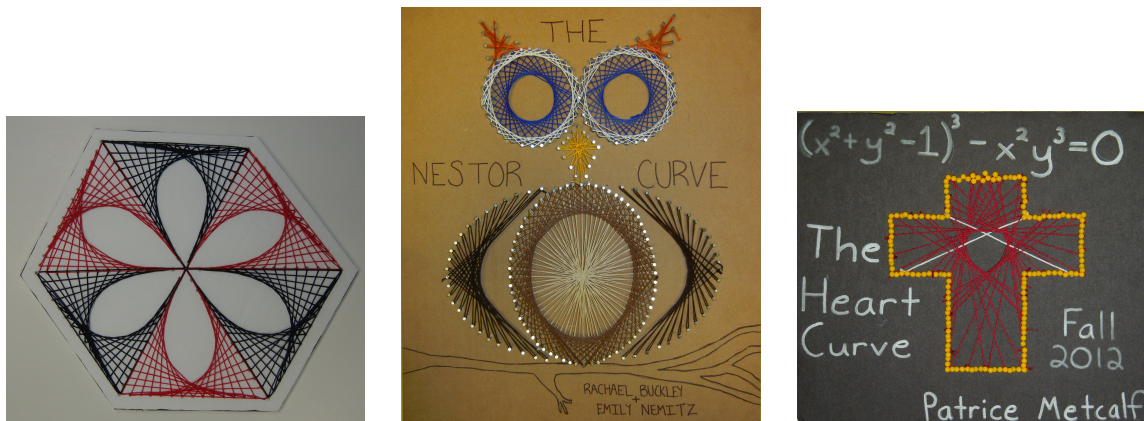

FIGURE 13. Original student string art.

7.7. Calculus. Rates and Calculus Anything that is a rate is linear by implication. In other words, lines are what tell us all about Calculus. So there needs to be big hooks to this book.

Points. Have no parts or joints. How can they combine. To form a line?

J.A. Lindon (; -)

Quadratic Growth and Problem Solving with Patterns

The tantalizing and compelling pursuit of mathematical problems offers mental absorption, peace of mind amid endless challenges, repose in activity, battle without conflict, refuge from the goading urgency of contingent happenings, and the sort of beauty changeless mountains present to senses tried by the present-day kaleidoscope of events.

Morris Kline (American Mathematician; 1908 - 1992)

It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts.

Paul Halmos (Hungarian born American Mathematician; 1916 - 2006)

Almost every American who has a degree, however ignorant, can live better than even competent people in much poorer countries around the world... But this cannot last long in the situation when "competence" and a diploma tautologically mean each other. The advantages enjoyed by Americans are the results of real competence and real efforts of previous generations... And someday ignorant people with degrees and diplomas may want power according to their papers rather than real competence. We Russians have some experience of this sort... It is clear to me right now that the winners in the modern world will be those countries which will really teach their students to think and solve problems. I sincerely wish America to be among these.

Andrie Toom (Russian Mathematician; 1942 -)

The problem is not that there are problems. The problem is expecting otherwise and thinking that having problems is a problem.

Theodore Rubin (American Psychiatrist; 1923 -)

The best way to escape from a problem is to solve it.

Alan Saporta (Musician; -)

We only think when confronted by a problem.

John Dewey (American Educator; 1859 - 1952)

When I am working on a problem, I never think about beauty. I only think of how to solve the problem. But when I am finished, if the solution is not beautiful, I know it is wrong.

Buckminster Fuller (American Architect; 1895 - 1983)

1. Introduction

We all face problems each and every day. What an amazing thing that brains have developed to help us solve some of these problems. It is a miraculous thing. Think of all of the problems that have been solved to make life as "simple" as it is. 5,000 years ago there was no metal. Now for a few days wages you can afford to wear your iPod onto an airplane that will fly at 50,000 feet at 600 miles per hour and take you to New York City, a city where over 8 million people live in an area of just over 300 square miles. The problem solving that has enable this to happen is immense.

So maybe you don't want to be somebody who thinks about how we solve problems: a philosopher, a psychiatrist, or an educational theorist. About now you might be bringing out the oft-used contemporary mantra, "When will I need to know this?" Show me the money! We hope that your work through the

material in these guides will help you develop an understanding of why this is an unfortunate and limiting attitude. Whether you have arrived at this point yet, will arrive there, or even differ with us - regarding learning in general and mathematics in particular - nobody can deny that they will always need to know how to solve problems. Each day brings forth a wealth of problems to solve. And there are many benefits to solving problems:

> The value of a problem is not so much coming up with the answer as in the ideas and attempted ideas it forces on the would be solver.
>
> **I.N. Herstein** (Polish born American Mathematician; 1923 - 1988)

> Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.
>
> **George Polya** (Hungarian Mathematician; 1887 - 1985)

> Solving problems is a practical art, like swimming, or skiing, or playing the piano if you wish to become a problem solver you have to solve problems.
>
> **George Polya** (Hungarian Mathematician; 1887 - 1985)

> The problems that exist in the world today cannot be solved by the level of thinking that created them.
>
> **Albert Einstein** (German Physicist; 1879 - 1955)

In working on arithmetic growth and piecewise arithmetic growth in Chapter **??** we developed several tools for examining patterns. In choosing the next topic, we faced several options. One option would be following the structure of the tools we used in examining these patterns and move on to the next type of growth which is called *quadratic growth*. If we did this then we would move on and do the subsequent types of growth, *cubic growth*, *quartic growth*, *quintic growth*, and so on. The problem with this is that while each of these types of growth is important, this pattern would continue on ad infinitum and would make for a very long book. The second option, which is the one we took, is to not focus on the specific structure of the tools we developed in working on arithmetic and piecewise arithmetic growth, but to look at similar problems that can be solved using generalized versions of the tools we have acquired.

2. Three Motivating Problems

Our work in this chapter will be motivated by three problems:

The Circle Problems: On a circle n points are drawn. Lines are drawn which connect each of the points to all of the other points on the circle. Circle Problem 1: How many lines must be drawn to connect all of the points? Circle Problem 2: Into how many regions is the circle decomposed by the lines?

The Line Problem: On a plane n lines are drawn. Each line intersects every line exactly once and no more than two lines intersect at a single point. Line Problem 1: How many points of intersection are formed by the lines? Line Problem 2: Into how many regions is the plane divided by the lines?

The Handshake Problem: A number of people are to be introduced to each other by shaking hands. If each person shakes hands with every person (excluding themselves) exactly once, what is the total number of handshakes that are made?
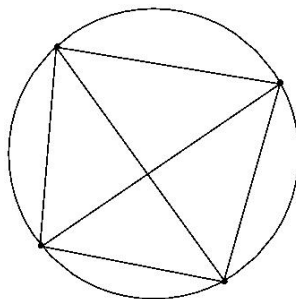
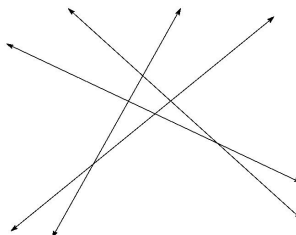FIGURE 1. A circle with four points, six lines and eight regions

FIGURE 2. Four lines with six points of intersection and eleven regions.

These are your problems to solve. You should spend significant time working on these problems before you move on. As wrestle with these problems, be reminded of what John Dewey told us:

> No thought, no idea, can possibly be conveyed as an idea from one person to another.
> When it is told it is to the one to whom it is told another fact, not an idea... Only by
> wrestling with the conditions of the problem at first hand, seeking and finding his own
> way out, does he think.

Indeed, the whole purpose of this series is to get you working on problems, to get you thinking and for you to be mathematically active. Here the problems are both clear enough and meaty enough that we can give them to you without much guidance. We are confident that you can make significant headway on these problems on your own.

3. Problem Solving Strategies

Here we provide some strategies that can be used to help solve the three problems in Section 2 and related problems. These strategies are presented in the same guided discovery framework that typical investigations are. In how much detail you decide to consider these sections depends on your success in finding patterns in the original problems, your success in completing the investigations, and the directions/requirements of your teacher.

Did you spend an hour working on the problems above? Have you solved them to your satisfaction? If not, go back and do this. This chapter will not be successful if you have not done this. And then what? It depends on you, your teacher, and the nature of your solutions to these problems. Our hope is that in your investigation of these problems you have found sufficiently robust ideas, strategies, relationships, and patterns that you can solve the problems in Section 4:**Using What You Have Discovered**. For now skip ahead to this section and try out some of these investigations. Yes, we said, SKIP AHEAD TO THE END. You'll know when and if you need to come back

So you're back. Yeah, we expected you might be. There is a reason for the topics and investigations in between. Your idea of "solving" these problems may not have been robust enough to help you solve all of the investigations at the end. Your strategies might be limited, allowing you to solve only certain problems in the final section. Etc. The intervening sections provide guided prompts for a variety of different strategies that are both typical and effective means of solving the three problems you've been working on. Maybe you need some helping solving one or more of the three problems - work through a section that may seem to be related to your strategy if you are stuck. Try one of these if you need a push. Work through some of these sections if your strategies did not apply to some of the investigations at the end and you need a new approach. Or just work through these intervening sections to see what you can discover there. It's up to you. (And perhaps your teacher.) Additionally, there are many ways that you can choose to work on these problems cooperatively with peers, including the wonderful Jigsaw Classroom method.

3.1. Problem Solving Strategy: Collecting Data. In each of the three problems in Section 2 there is a variable - the number of people, points, and lines - which is indeterminate. You need to solve the problem no matter how many of each they are. A natural response to this is to collect some data and see what this quantitative data might tell you.

Let's do that here.

We will begin with the Circle Problems. In Figure 3 are circles with two, three and four points on the circle. These points are connected by straight lines as shown.
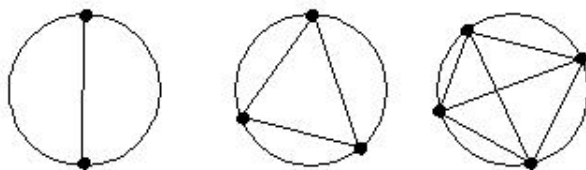
FIGURE 3. Three circles with points and lines.

With n, P, R and L representing the stage number, number of points, number of regions, and number of lines respectively, the properties of the figures can be tabulated as Table 1:

n	P	L	R
1	2	1	2
2	3	3	4
3	4		
4			
5			

TABLE 1. Table of values for the properties of the figures in Figure 3

1. Determine the values of L and R for the 3rd stage.
2. Assuming the pattern continues in the indicated way, draw the 4th stage in this pattern.
3. Use Investigation **2** to determine the values of P, L and R for the 4th stage.
4. What should the be the values of P, L and R for the 5th stage. Explain.

30

5. Now draw the 5$^{\text{th}}$ stage in the pattern.
6. Use Investigation **5** to determine the values of P, L and R for the 5$^{\text{th}}$ stage.
7. Do your predictions for the values of P, L and R when $n = 5$ in Investigation **4** match with the actual values of P, L and R in Investigation **6** when $n = 5$?
8. Make a conjecture which describes how we can predict the value of P as an explicit function of the stage number n.
9. Find and describe, in words, a pattern in the values of L as a function of the stage number n. Is it easy to find an explicit equation which describes L as a function of n? Explain.

Now let's switch to the lines problem.

10. Draw two lines, extending indefinitely, that are neither concurrent nor parallel.
11. How many points of intersection are there in the pair of lines you drew in Investigation **10**?
12. Draw three lines, extending indefinitely, so that no pair are concurrent, no pair are parallel, and only two lines cross at any point of intersection.
13. How many points of intersection (where two lines meet) are there among the three lines you drew in Investigation **12**?
14. Repeat Investigation **12** for four lines.
15. How many points of intersection (where two lines meet) are there among the four lines you drew in Investigation **14**?
16. There is a pattern in your answers to Investigations **11**, **13**, and **15**. Describe it.
17. Use Investigation **16** to make a conjecture about the number of points of intersection if you draw five lines, six lines, and seven lines.

Now let's try the handshake problem where you might want to have a small group of people nearby to test things out.

How many handshakes will there be when there are

18. two people in the room?
19. three people in the room?
20. four people in the room?
21. five people in the room?
22. Complete the table of values in Table 2, where n is the number of people and h is the total number of handshakes. You should see a pattern forming. Describe the pattern in detail.

n	h
1	
2	
3	
4	
5	
6	
7	
8	
9	

TABLE 2. Table of values for the number of handshakes with n people.

23. How is this problem related to the problems considered in Investigations **1-17**? Explain in detail. In particular, if you were to make tables for the data of these earlier problems, how would they compare?

In the next few sections we will consider several different strategies for solving problems like the ones in Investigation **23**. To help you understand these methods we will solve each of the problems using that method. You should not view this as busy work, but rather an opportunity to understand how these methods work using a familiar context.

3.2. Recognizing When You Are Faced With One of These Types of Problems. In the last chapter we saw that the first differences of the terms in a numerical sequence are constant precisely when the sequence is generated by a linear equation of the form $f = m\,n + b$. In Discovering the Art of Mathematics: Calculus in this series we extend this result to a much more general pattern at the heart of a discrete calculus:

Theorem 1. *[Fundamental Theorem of Discrete Calculus] Let S be a sequence. The sequence of kth differences is the lowest degree of constant differences if and only if the original sequence is generated by a kth degree polynomial function.*

We can use this profitably to solve problems like those considered in Sections 3.1-3.5

24. In Table 3 is data from Investigation **22**. Fill in the rest of the data for h and the first differences.

n	h	$1^{st}\Delta$
1	0	
		$>$ 1
2	1	
		$>$ 2
3	3	
		$>$ 3
4	6	
		$>$
5		
		$>$
6		
		$>$
7		
		$>$
8		
		$>$
9		

TABLE 3. Table of values for the number of handshakes with n people and the first differences.

25. Does this data belong to a linear equation? Explain.

We can now take the differences of the values in the column labeled $1^{st}\Delta$, these are called the **second differences**. The second differences are traditionally denoted by $2^{nd}\Delta$.

26. In Table 4 determine the second differences for the handshake problem.

32

n	h	$1^{\text{st}}\Delta$	$2^{\text{nd}}\Delta$
1	0		
		> 1	
2	1		> 1
		> 2	
3	3		> 1
		> 3	
4	6		>
		>	
5			>
		>	
6			>
		>	
7			>
		>	
8			>
		>	
9			

TABLE 4. Table of values for the handshake problem with the first and second differences.

27. What does Investigation **26** and Theorem 1 tell you about the type of function that generates the handshake data?

3.3. Problem Solving Strategy: Gauss's Epiphany. Carl Friedrich Gauss (German Mathematician; 1777 - 1855) is, without doubt, one of the greatest mathematicians that ever lived. His work is explored in detail in Discovering the Art of Mathematics: Number Theory in this series. This strategy and group of explorations is named after an epiphany widely attributed to him (although most likely apocryphal):

> The story goes that while a student in elementary school, his teacher gave the class the task of adding up all the numbers from 1 to 100. The teacher had scarcely finished giving out the assignment when Gauss announced that he was done and that the sum was 5050.

Figure 4[1] illustrates Gauss's method.

28. Explain what Gauss noticed and why his method gives the correct answer for the sum

$$1 + 2 + 3 + \cdots + 100.$$

29. Show that we can determine the number of lines needed to connect 5 points around a circle according to the approach in Section 3.1 by computing a sum similar to the one Gauss had to do.

30. Show that we can determine the number of points of intersections when 7 lines are drawn in the plane according to the approach in Section 3.1 by computing a sum similar to the one Gauss had to do.

31. Show that we can determine the number of handshakes for 9 people by computing a sum similar to the one Gauss had to do.

---

[1]From The Joy of Mathematics: Discovering Mathematics All Around You by Theoni Pappas, Wide World Publishing/Tetra, 1986.

$$1 + 2 + 3 + \ldots + 50 + 51 + \ldots + 98 + 99 + 100$$

101

101

101

101

50 x 101= 5050

FIGURE 4. Gauss's method for computing the sum $1 + 2 + 3 + \cdots + 100$

32. Use Gauss's method to compute the sum in Investigation **29**. Compare this answer to your answer in Investigation **3**, does this method give the correct number of lines?

33. Use Gauss's method to compute the sum in Investigation **30**. Compare this answer to your answer in Investigation **17**, does this method give the correct number of points of intersection?

34. Use Gauss's method to compute the sum in Investigation **31**. Compare this answer to your answer in Investigation **22**, does this method give the correct number of handshakes?

35. Use this method to determine how many handshakes there are if there are 27 people in a room.

36. Use this method to determine how many points of intersection there are if 53 lines in the plane are drawn according to the approach in Section 3.1.

37. Use this method to determine how many lines are needed to connect 85 points around a circle according to the approach in Section 3.1.

Each sum in Investigations **35-37** involved odd values of n so the value of $n - 1$ used in Investigation **42** is even. However there is a potential difficulty is when n is even.

38. Write out the sums needed to compute the number of handshakes when $n = 6, 8$ and 10, and then try using Gauss's method to compute the sums. What is the difficulty when n is even?

39. Find a way to resolve this difficulty and precisely describe your resolution. Be sure to check that your resolution gives the correct answer for the number of handshakes for $n = 6, 8$ and 10.

40. Use Gauss's method to determine the sum of the series

$$1 + 2 + 3 + \cdots + 1,345,217 + 1,345,218 + 1,345,219?$$

Each of the problems Investigations **35-40** involve determining the value of an arithmetic series of the form:

$$1 + 2 + 3 + \cdots + (n - 2) + (n - 1)$$

41. Use Gauss's method to determine a formula for the series $1 + 2 + 3 + \cdots + (n - 2) + (n - 1)$ as a function of the variable n. That is, use Gauss's method to find a formula that allows to find the sum of $1 + 2 + 3 + \cdots + (n - 2) + (n - 1)$ with only knowing the value of n.

42. Check that the formula in Investigation **41** provides the correct answers for Investigations **35-40**.

43. What do you think of this method?

34

3.4. Problem Solving Strategy: Blocks and Other Manipulatives. Another strategy involves using a physical model to solve the problems like the ones we have considered. In Figure 5, a student represents the total number of handshakes using ***Multilink cubes*** -plastic cubes that join together.
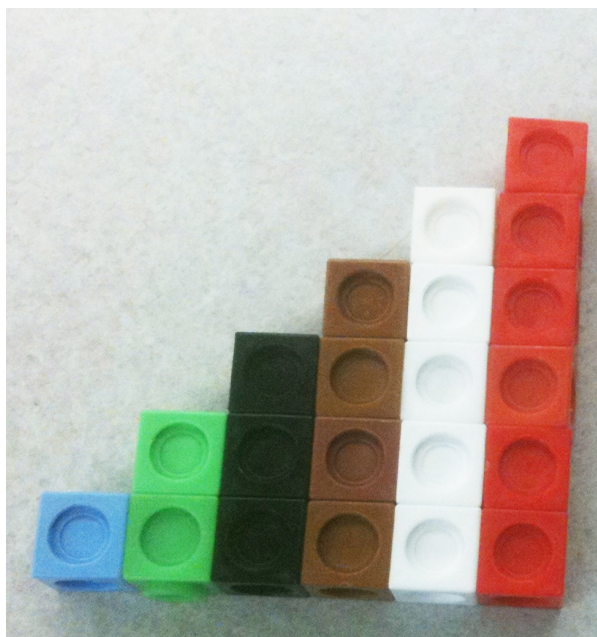
FIGURE 5. Staircase model for computing the number of handshakes with 7 people.

As shown in Figure 6, she then determines the value of $1 + 2 + 3 + 4 + 5 + 6$ by creating a second identical staircase out multiink cubes and joining the two together to form a 7×6 rectangle. She then uses that rectangle to show that $1 + 2 + 3 + 4 + 5 + 6 = \frac{7 \times 6}{2} = \frac{42}{2} = 21$.[2]
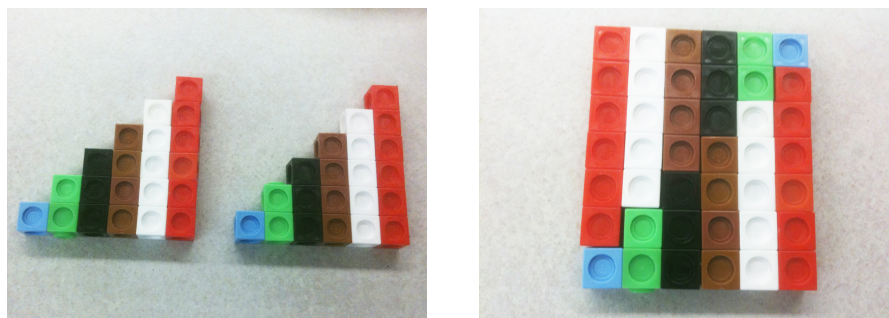
FIGURE 6. Staircase model for computing the sum $1 + 2 + 3 + 4 + 5 + 6 = \frac{7 \times 6}{2} = \frac{42}{2} = 21$

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[2]From Essentials of Mathematics: Introduction to Theory, Proof, and the Professional Culture by Margie Hale, Mathematical Association of America, 2003.

44. Explain why the dimensions of the rectangle formed by joining the two staircases together in Figure 6 is 7×6.

45. Draw a staircase that represents the number of lines needed to connect 5 points around a circle according to the approach in Section 3.1.

46. Draw a staircase that represents the number of points of intersections when 7 lines are drawn in the plane according to the approach in Section 3.1.

47. Draw a staircase that represents the number of handshakes for 9 people.

48. Use the staircase method described above to compute the sum in Investigation **45**. Compare this answer to your answer in Investigation **3**, does this method give the correct number of lines?

**49.** Use the staircase method described above to compute the sum in Investigation **46**. Compare this answer to your answer in Investigation **17**, does this method give the correct number of points of intersection?

50. Use Gauss's method to compute the sum in Investigation **47**. Compare this answer to your answer in Investigation **22**, does this method give the correct number of handshakes?

51. Use the staircase method to determine how many handshakes there are if there are 27 people in a room.

52. Use the staircase method to determine how many points of intersection there are if 53 lines in the plane are drawn according to the approach in Section 3.1.

53. Use the staircase method to determine how many lines are needed to connect 85 points around a circle according to the approach in Section 3.1.

54. Use the staircase method to determine the sum of the series

$$1 + 2 + 3 + \cdots + 1,345,217 + 1,345,218 + 1,345,219.$$

Each of the problems Investigations **51-54** involve determining the value of an arithmetic series of the form:

$$1 + 2 + 3 + \cdots + (n - 2) + (n - 1)$$

55. Use the staircase method to determine a formula for the series $1 + 2 + 3 + \cdots + (n - 2) + (n - 1)$ as a function of the variable n. That is, use the staircase method to find a formula that allows to find the sum of $1 + 2 + 3 + \cdots + (n - 2) + (n - 1)$ with only knowing the value of n.

56. Check that the formula in Investigation **55** provides the correct answers for Investigations **51-54**.

57. What do you think of this method?

3.5. Problem Solving Strategy: Combinatorics - The Art of Counting. The important online mathematical encyclopedia mathworld.wolfram.com defines combinatorics as follows: Combinatorics is the branch of mathematics studying the enumeration, combination, and permutation of sets of elements and the mathematical relations that characterize their properties. More colloquially, combinatorics is the mathematical art and science of counting. It is a critical tool in many areas of mathematics and relies heavily on patterns. This is exactly what we need to solve the handshake problem since it is all about counting.

Here's an example of combinatorial reasoning. A town sports league has each team play every other team exactly twice, once as the home team and once as the visiting team. How many games must they schedule? With 2 teams there are clearly only 2 games. With 3 teams you can check that there are 6 games. And with 4 teams there are 12 games. What if there were 24 teams? It seems complicated. But we can reason as follows. As a home team, each team must play 23 games, one with each of the teams in the league. Since this is true for each team and there are 24 teams, there are $23 \times 24 = 552$ games.

58. Describe precisely what's wrong with the following argument for determining the number of hand-shakes with 9 people in the room: Each person must shake hands with 8 other people. Since there are 9 people, there are $9 \times 8 = 72$ handshakes.

59. Despite it's incorrectness, the method in Investigation **58** can be adapted to provide the appropriate number of handshakes when there are 9 people in a room. Explain.

60. Use the combinatorics method from Investigations **58-59** to determine the number of lines needed to connect 5 points around a circle according to the approach in Section 3.1. Compare this answer to your answer in Investigation **3**, does this method give the correct number of lines?

61. Use the combinatorics method from Investigations **58-59** to determine the number of points of intersections when 7 lines are drawn in the plane according to the approach in Section 3.1. Compare this answer to your answer in Investigation **17**, does this method give the correct number of points of intersection?

62. Use the combinatorics method to determine the number of handshakes if there are 27 people in a room.

63. Use the combinatorics method to determine the number of handshakes if there are 532 people in a room.

64. Suppose that 53 lines in the plane are drawn according to the approach in Section 3.1. Use the combinatorics method to determine the total number of points of intersection.

65. Suppose 85 points around a circle are connected with lines according to the approach in Section 3.1. Use the combinatorics method to determine the total number of lines.

66. Use the combinatorics method to determine to determine a formula for the number of handshakes as a function of the number, n, of people in a room. That is, find a formula that allows to find the sum of $1 + 2 + 3 + \cdots + (n - 2) + (n - 1)$ with only knowing the value of n.

67. What do you think of this method?

3.6. Problem Solving Strategy: Discrete Calculus. The type of function you identified in Investigation **27** may sound intimidating, but you have encountered this type of function before. You should remember the *quadratic formula* from high school algebra. It is used to calculate the roots of the general ***quadratic function***, $f = a\,n^2 + b\,n + c$. This function is also known as the general ***2nd degree polynomial function***. You only need to determine the values of a, b, and c. Before we return to the data from the handshake problem, we will work through a method to find the values of a, b, and c for another set of data that describes a quadratic function. Investigations **68-72** will all refer to the data given in Table 5.

Since the second differences in Table 5 are constant, we know this data must be described by a quadratic function $f = a\,n^2 + b\,n + c$.

68. Substitute $n = 0$ and the corresponding value for f into the quadratic and solve for c.

69. Substitute $n = 1$ and the corresponding value for f into the quadratic to generate an equation containing only the variables a and b.

70. Repeat Investigation **69** with $n = 2$.

71. Solve the equations in Investigations **69-70** simultaneously to determine the values of a and b.

72. Write out the quadratic explicitly and show that it correctly generates the appropriate values for $n = 0, 1, 2, 3, 4, 5, 6$.

73. Use the ideas from Investigations **68-71** to determine appropriate values of a, b, and c for the (quadratic) handshake function described by the table in Investigation **26**. **Hint:** To make

n	f	$1^{\text{st}}\Delta$	$2^{\text{nd}}\Delta$
0	4		
		> 8	
1	12		> 4
		> 12	
2	24		> 4
		> 16	
3	40		> 4
		> 20	
4	60		> 4
		> 24	
5	84		> 4
		> 28	
6	112		

TABLE 5. Table of values for a quadratic function.

finding the values of a, b and c easier, you should add a row to the table in Investigation **26**; when there are no people in the room (i.e $n = 0$), how many handshakes are there?

74. What do you think of this method?

3.7. Problem Solving Strategy: Factoring. Often we cannot see patterns simply because there are several intertwined processes at work that can only be understood once they are isolated. For number sequences, factoring can often help us untangle these processes so we can describe the underlying processes that give rise to the pattern.

75. Let's consider the function defined by the pattern in Table 6

n	f	Factors of f
0	6	1×6 or 2×3
1	12	1×12 or 2×6 or 3×4
2	20	1×20 or 2×10 or 4×5
3	30	
4	42	
5	56	
6	72	

TABLE 6. Table of values, with factors, for a quadratic function.

Fill in the remaining factors of f that have not been filled in.

76. Choosing appropriate factors from each row, there is a very regular pattern to the factors. Highlight these factors and describe the pattern precisely.

77. Describe the pattern of smaller factors from Investigation **76** as a function of n.

78. Describe the pattern of larger factors from Investigation **76** as a function of n.

79. Suitably combine Investigations **77-78** to determine an explicit formula for the pattern f as a function of n. Check that this function provides the appropriate data for $n = 0, 1, 2, 3, 4, 5, 6$.

38

We'd like to use this approach for the handshake problem as well.

80. Use Investigation **22** to fill in the table inTable 7 and then determine factors of the data for h in the middle column.

n	h	Factors of h
2		
3		
4	6	2×3
5		
6		
7		
8		
9		
10		
11		

TABLE 7. Table of values, and factors, for the number of handshakes with n people.

81. You should see a clear pattern among some subset of the factors. Describe it precisely. **<u>Hint:</u>** Consider the factors when n is odd.

While pattern you described in Investigation **81** works well for a subset of the factors, but it is not clear what the pattern should be for all the factors in Investigation **81**. It is easier to describe if we force a factor of $\frac{1}{2}$ into each set of factors. In other words, instead of $6 = 2 \times 3$ we write $6 = \frac{1}{2} \times 4 \times 3$.

82. Rewrite the table in Investigation **80** by forcing a factor of $\dfrac{1}{2}$ into each of the factors you have. Precisely describe the pattern in the factors that you now see.

83. Following the examples of Investigations **77-79**, determine an explicit formula for h as a function of n.

84. Compare your expression for h in Investigation **83** with the expressions for h you obtained using the previous methods.

3.8. Problem Solving Strategy: Recursion. One meaning of the word *recur* is to happen again. While we have not defined exactly what constitutes a pattern, there is something inherent in our understanding of this term that in a pattern something happens again, and again, and again,... Mathematicians have adopted this root and use the word **_recursion_** to name a process in which objects are defined relative to prior objects in the same process. They also use the term *recursive* as the associated adjective.

While you may not have heard the term before, the importance of recursion in the world around us cannot be understated. Populations, weather, account balances, and many other real phenomena we might study are dependent at any stage on their size, behavior, distribution, and makeup at prior stages. Anybody who has used a spreadsheet has used recursion when they define a new cell using information in other cells. Recursion underlies the development of fractals and chaos as described in <u>Discovering the Art of Mathematics: Geometry</u>.

In an important sense, to discover a recursive relationship is to really see what it is that makes the object being studied a pattern. They can also be helpful in problem solving.

Investigations **85-95** refer to the table of values and first differences shown in Table 8.

n	f	$1^{\text{st}}\Delta$
0	2	
		$>$ 2
1	4	
		$>$ 4
2	8	
		$>$ 6
3	14	
		$>$ 8
4	22	
		$>$ 10
5	32	
		$>$ 12
6	44	

TABLE 8. Table of values and the first differences.

We are looking for a recursive relationship in this pattern; that is, we are trying to find a pattern that allows us to find a way of describing the nth stage from the $(n-1)^{\text{st}}$ stage. To do this it will be more convenient for us to use the sequence notation for f.

85. Express the values for f in Table 8 in the sequence notation for $n \leq 6$. That is, $f_0 =$___, $f_1 =$___, $f_2 =$___, etc.

86. What is the amount we add to f_0 to get f_1?

87. ... to f_1 to get f_2?

88. ... to f_2 to get f_3?

89. ... to f_3 to get f_4?

90. ... to f_{11} to get f_{12}?

91. Based on your answers to Investigations **86-90**, what is the amount we add to f_{n-1} to get f_n? Explain.

92. Use your answer to Investigation **91** to write down an equation that expresses how we can determine the value of f_n from f_{n-1}.

The equation you came up with in Investigation **92** is called a ***recursive definition*** for f. As long as we have the equation and a starting value with which to begin, called the ***initial value***, (which in this case would be $f_0 = 2$) we can find the value of f_n for any value of n.

93. Use your answer to Investigation **92** to determine the values of f_n for $n = 7$ through $n = 15$.

94. Would this method be useful if you had to determine f_{100}? Explain.

95. Use the ideas from Section 3.6 to find an explicit formula for f as a function of n, and use this formula to compute the value of f for $n = 100$. Is this easier than using the recursive definition to find the value when $n = 100$? Explain.

96. Despite the fact that recursion can be less efficient than an explicit formula, give some reasons why you think recursion might have been developed and is still used extensively.

97. A joke among mathematicians is that a dictionary had the following definition:

recursion n \ri-'k r-zh n\ n See "recursion".

Explain this joke. To be funny jokes generally have some kernel of truth in them. What underlying truth does this joke point out about languages and/or dictionaries?

We now want to apply these ideas to the handshake problem.

98. Use Investigation **22** to fill in the table in Table 9 and then determine the first differences.

n	h	$1^{st}\Delta$
1		
2		
3		
4		
5		
6		
7		

TABLE 9. Table of values, and first differences, for the number of handshakes with n people.

99. Use the ideas from Investigations **85-92** to find the recursive definition for h.

4. Trying Out What You Have Discovered

Now comes your chance to utilize the strategies you have discovered and/or learned. You should be able to answer each of the questions in this section. If not, you should go back and work to find a method that you can use. Also, please note that most problems can be solved using several different methods.

100. How many handshakes are there if there are 27 people in a room?

101. How many handshakes are there if there are 532 people in a room?

102. Determine a closed-term, algebraic expression for the number of handshakes (represented by the dependent variable h) there are if there are n people in a room.

103. Determine a recursive relationship for the number of handshakes when there are n people in the room (represented functionally by h_n) as a function of the numbers of handshakes with fewer people in the room (denoted by h_{n-1} respectively).

104. How many points of intersection are there if 53 lines as described in the Line Problem?

105. How many points of intersection are there if 264 lines as described in the Line Problem?

106. Determine a closed-term, algebraic expression for the number of points of intersection (represented by the dependent variable p) there are if there are n lines drawn as described in the Line Problem.

107. Determine a recursive relationship for the number of points of intersection (represented functionally by p_n) as a function of the number of points of intersection when there are fewer lines (denoted by respectively by p_{n-1}).

108. How many lines are needed to connect 85 points around a circle as described in the Circle Problem?

109. How many lines are needed to connection 388 points around a circle as described in the Circle Problem?

110. Determine a closed-term, algebraic expression for the number of lines needed (represented by the dependent variable L) if there are n points connected as described in the Circle Problem.

41

111. Determine a recursive relationship for the number of lines needed (represented functionally by L_n) as a function of the numbers of lines needed with fewer points around a circle (denoted by L_{n-1} respectively).

Investigations **112-114** concern the series $1 + 3 + 5 + \cdots + 2101 + 2103 + 2105$.

112. While we dealt with some series Section 3 this one differs from those in a very important way. **Hint:** Look at the first differences.

113. To use the strategies in Section 3 we need to be able to figure out the number of terms in the series $1 + 3 + 5 + \cdots + 2101 + 2103 + 2105$. Use the techniques from Section **??** to find a formula for the terms in this series; and then use this formula to determine the number of terms in the series.

114. Use your answer to Investigation **113** and the ideas in Section 3 to determine the sum of the series $1 + 3 + 5 + \cdots + 2101 + 2103 + 2105$.

115. Use the strategy from Investigations **112-114** to determine the value of the sum of the following series: $1 + 4 + 7 + \cdots + 158509 + 158512 + 158515$.

116. Determine the value of the sum of the following series: $5 + 10 + 15 + 20 + \cdots + 7,895$.

117. Determine the value of the sum of the following series: $8 + 10 + 12 + \cdots + 11212 + 11214 + 11216$.

118. Determine the value of the sum of the following series: $284 + 291 + 298 + 305 + \cdots + 3056 + 3063 + 3070$.

119. Each of the series in Investigations **112-118** are called ***arithmetic series***. Explain why we call these arithmetic series.

In general, arithmetic series can be written as:
$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 2)d) + (a + (n - 1)d) + (a + n\,d)$$

120. Determine a closed-term, algebraic function for the sum of this series as a function of the parameters a, d, and n.

121. Use the formula in Investigation **120** to check your answers to Investigations **112-118**.

As long ago as ancient Greek mathematicians such as Pythagoras (circa 580 - 500 BC), and probably longer, people looked at patterns of numbers created by shapes. Some of these, the so-called ***figurate numbers***, are illustrated in Figure 7.

122. Show that the numbers formed by the stages in the shape of a triangle, the ***Triangular Numbers***, are the same as the Handshake Numbers.

123. Show that the numbers formed by the stages in the shape of a square, the ***Square Numbers***, are 1, 4, 9, 16, 25, ... as we might expect.

When we generated the handshake numbers in Section 3.3, we did this as sums of series. In this series the first difference between consecutive ***terms*** (i.e. the numbers we are adding up in the series) is always 1. Namely,

$$1 = 1$$
$$1 + 2 = 3$$
$$1 + 2 + 3 = 6$$
$$1 + 2 + 3 + 4 = 10$$

124. Show how the square numbers can be written as sums of series.

125. What are the first differences between the terms that make up the series in Investigation **124**?

126. You should see a pattern forming. Show that the Pentagonal Numbers follow this pattern.
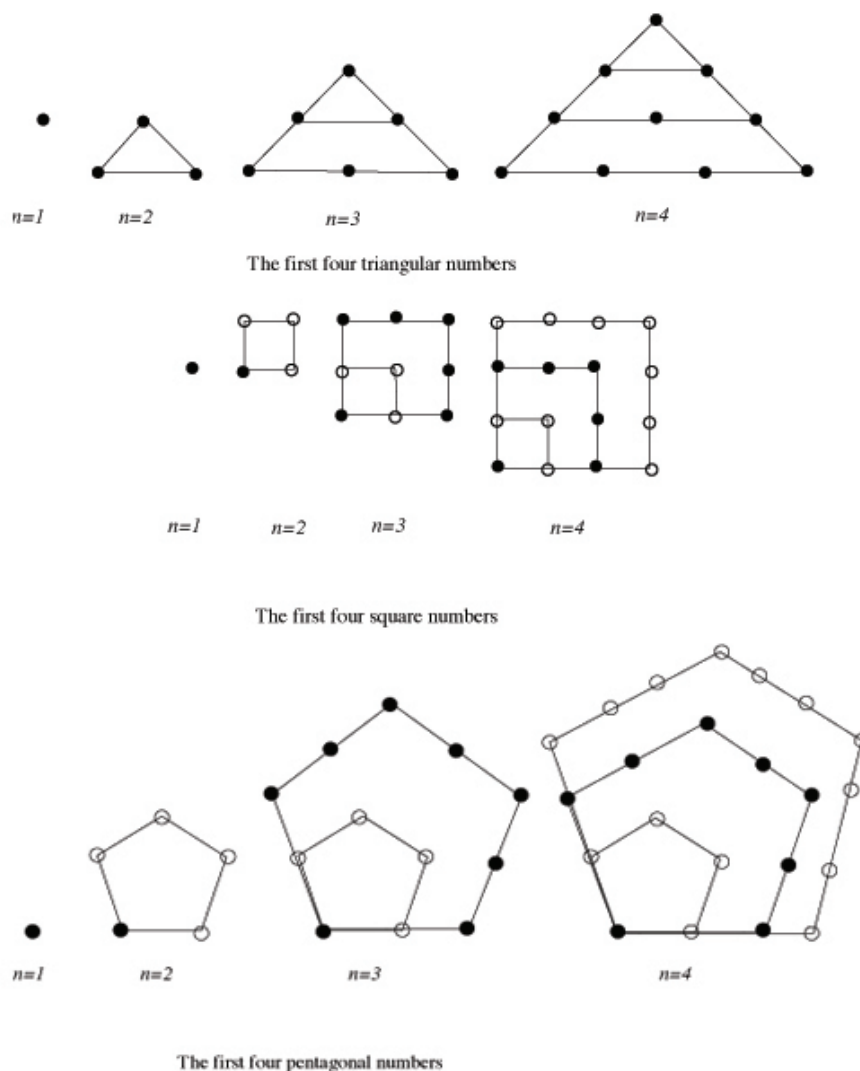
The first four triangular numbers

The first four square numbers

The first four pentagonal numbers

FIGURE 7. Figurate Numbers

127. Use the ideas from Section 3 to determine a formula for the nth Pentagonal Number.

128. Use your observation from Investigation **126** to create the *Hexagonal Numbers*.

129. Use the ideas from Section 3 to determine a formula for the nth Hexagonal Number.

Perhaps surprisingly, the Natural Numbers 1, 2, 3, 4, 5, can also be formed in this way:

130. Show how the natural numbers can be written as sums of series where the terms are constant with value 1.

131. Using Investigation **130** and the geometric patterns investigated above to explain visually why it makes sense to call the natural numbers the *Linear Numbers*.
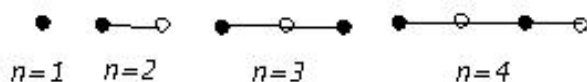
The first four Linear Numbers

FIGURE 8. The Linear Numbers

5. Essays

132. John Dewey (American Educator; 1859 - 1952) once wrote the following:

> No thought, no idea, can possibly be conveyed as an idea from one person to another. When it is told it is to the one to whom it is told another fact, not an idea... Only by wrestling with the conditions of the problem at first hand, seeking and finding his own way out, does he think.

Now that you have completed this section on problem solving, do you believe that you have a better, or worse, understanding of the techniques in this section than you would have if we had just told you how to solve these problems? Explain.

133. Why were there four problems at the outset?

134. We expect that you used several different strategies to solve the problems in Section 4. If you look back at the strategies that were described in Section 3 there were eight strategies. Is there some value in having so many different strategies? Explain.

6. Further Investigations

The Line Problem can be refined and extended in many ways. Beautiful patterns continue to emerge and the problems range from the level of the Line Problem to areas of open research questions.

F1. Repeat your analysis of the Line Problem by determining how many unbounded regions are formed by the lines.

F2. Repeat your analysis of the Line Problem by determining how many bounded regions are formed by the lines.

F3. Among the bounded regions, different shapes may be formed. Can you determine the types and number of each of these shapes?

Instead of allowing our lines to be placed arbitrarily, we can arrange the lines regularly so they form regular polygons in their center, as shown below.[3]

Insert p. 118 Figure 11.1 from Pedersen.

F4. Into how many unbounded regions is the plane divided by the lines?

F5. Into how many bounded regions is the plane divided by the lines?

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[3]Adapted from "Platonic Divisions of Space" by Jean Pedersen in Mathematical Adventures for Students and Amateurs.

CHAPTER 5

The Use of Patterns and Language in the Creation of Powerful Number Systems

Science is the attempt to make the chaotic diversity of our sense-experiments correspond to a logically uniform system of thought.

Albert Einstein (German Physicist; 1879 - 1955)

We encounter patterns all the time, every day: in the spoken and written word, in musical forms and video images, in ornamental design and natural geometry, in traffic patterns, and in objects we build. Our ability to recognize, interpret, and create patterns is the key to dealing with the world around us.

Marjorie Senechal (American Mathematician; 1939 -)

1. Names of Numbers

For each of the numbers whose name in English is given below, write the number named as a base-ten number:

135. Nineteen thousand, four hundred sixty-five.

136. Three hundred fifty two thousand, eight hundred nineteen.

137. Seventeen million, forty three thousand, five hundred eighty two.

When writing numbers in English one uses commas only to separate words as you would when writing the digits - only in groups of three. The word "and" is used to indicate where a decimal point goes.

For each of the numbers written below in base-ten, give their English name:

138. 784.

139. 562,978.

140. 6,587,581.

141. 5,914,490,937.

We can write large numbers, in fact as large as we desire, because the base-ten Hindu-Arabic number system that we use is positional. We do not need to adapt this system in any way, we just fill in appropriate digits to build larger and larger numbers. Such a number system is dramatically more powerful and flexible than the Greek, Roman, and Egyptian systems which were the dominant systems through much of recorded Western history.

This system that we generally take for granted is a profoundly powerful human language.

You should remember how this positional number system works. 5,327 is five thousands, three hundreds, two tens, and 7 ones, written in *expanded notation* as:

$$5 \times 1000 + 3 \times 100 + 2 \times 10 + 7 \times 1.$$

The *digits* are always 0 - 9 and the critical bases are ..., 1000, 100, 10, and 1.

For positive integers a and n we define a to the **power** n by $a \times a \times \cdots \times a$ where there exactly n factors of a in the product. The number a is called the **base** and the number n is called the **exponent**. It is then natural to call the numbers $a, a^2, a^3, \ldots$ the **powers of** a.

142. Is 100 a power of 10? Explain.

143. Is 1000 a power of 10? Explain.

144. Is 10000 a power of 10. Explain.

145. You should see a pattern forming in Investigations **142-144**. Use this pattern to write the number $1\underbrace{000\ldots000}_{n}$ as a power of 10 where the number has n zeroes following the lone digit 1.

146. Use Investigation **145** to explain why it is appropriate to call our number system the "base-ten" number system.

We can now simplify the expanded notation using what we have learned about powers of ten. So, for example, we can write $5,327$ as $5 \times 10^3 + 3 \times 10^2 + 2 \times 10^1 + 7 \times 10^0$ where, for the moment, we have defined $10^0 = 1$. (Shortly we will see that this really is a natural consequence of a pattern.)

147. Write the number in Investigation **138** in expanded notation using powers of ten.

148. Write the number in Investigation **139** in expanded notation using powers of ten.

149. Write the number in Investigation **140** in expanded notation using powers of ten.

150. Write the number in Investigation **141** in expanded notation using powers of ten.

Because successive powers of ten are a factor of ten larger than the power which precedes it we have a powerful tool to study things that exist on massive scales - like our universe. In their wonderful book Powers of Ten authors **Philip Morrison** (American Physicist and Author; 1915 - 2005)[1] and **Phylis Morrison** (American Teacher, Educator, and Author; - 2002) provide a tour of the universe by starting at the edge of our local cluster of galaxies and with each successive page moving our viewpoint a factor of ten closer. After some 25 pages we see we have been focusing on a couple lying on a blanket at a city park in Chicago, Illinois. Not stopping there, the photos continue to move ten times closer, eventually reaching the sub-atomic particles that make up the DNA of one of these people. Subsequent movies, flip-books, screen savers, and interactive Internet sites immortalize this powerful idea.[2]

The American Museum of Natural History in New York, New York integrates these ideas into a spectacular installation called "Scales of the Universe." At the center of this installation housed in the Rose Center for Earth and Space is the Hayden Planetarium - a 150 foot tall sphere which houses a full IMax theatre in the top half and an interactive tour on the bottom half. Spiraling around the Hayden Sphere is "Scales of the Universe" - a walkway through the sizes and scales of the universe. Instead of using visual images like Powers of Ten, it uses physical models which are successively compared to the massive Hayden Sphere which hangs right in front of your view to help you understand the awesome scale of the universe through the subatomic workings of each little piece of the universe.

What about naming really large numbers using English words?

151. Choose a number which is a four digit number when written in base-ten. Write this number in base-ten and then name this number using English words.

152. Now add a digit to the front and name this number using English words.

153. Continue adding digits one at a time to the front of your number and naming your new number using English words. Whenever possible, you need not name the number completely if a significant part of the name stays the same. Just note what part stays the same. You should continue doing this until you have a 14-digit number.

154. Suppose you were asked to add one more digit. Would you need any additional information to know how to name your number using English words? Explain.

155. Suppose you were asked to add four more digits. Would you need any additionally information to know now to name your number using English words? Explain.

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[1]More info here about him? E.g. SETI patriarch...

[2]This is also in The Very Large chapter. How do we reference it here again?

The terms million, billion, trillion, quadrillion, quintillion, sextillion, septillion, octillion, and non-illion were introduced by **Nicholas Choquet** (; -) in 1484 and appeared in print in a 1520 book by **Emile de la Roche** (; -). The meanings of these words were subsequently changed and there continue to be linguistic debates and discrepancies about the names of large numbers.

Nonetheless, there is a fascination with naming large numbers. Mathematicians have continued to develop different schemes. In 1996 **Allan Wechsler** (; -), **John Horton Conway** (; -), and **Richard Guy** (; -) proposed a system that can be extended indefinitely to provide an English word name for *any* number! We will consider this naming scheme now.

156. Complete the chart below, using your knowledge of prefixes and patterns to help you establish a pattern that gives meaning to the terms you are not certain of:

Name of Number	Number in Base-Ten	Number as Base-Ten Exponent
Thousand	1,000	
Million	1,000,000	
Billion		
Trillion		
Quadrillion		
Quintillion		
Sextillion		
Septillion		
Octillion		
Nonillion		

157. Do these names give you what you needed to positively answer Investigation **154** and Investigation **155**? Explain.

158. Using these names, how high can you count before you will not be able to name a specific number? Describe this number.

Our goal is to use patterns in this chart to expand our linguistic ability to name increasingly larger numbers.

159. Add two more columns to your table from Investigation **156** - the columns which have been started below. Complete the columns in the natural way.

Prefix of Number	Name Ordinal illion
None	zeroeth
mi	first
bi	second

160. Find patterns in the table that allow you to determine the 10th, 13th, and 21st illion as base-ten exponents.

161. Extending Investigation **160**, the nth illion is equal to $10^?$.

162. Conversely, what illion is 10^{57}? 10^{219}? 10^{399}?

It is clear how our base-ten exponents and ordinal illions can continue indefinitely. What we need to extend the English names of the numbers is more prefixes. Wechsler, Conway and Guy provided a way to extend these prefixes indefinitely. Their scheme relies on the following prefixes:

	Units	Tens	Hundreds
1	mi	deci^n	centi^{nx}
2	bi	viginti^{ms}	ducenti^n
3	tre*	triginta^{ns}	trecenti^{ns}
4	quad	quadraginta^{ns}	quadringenti^{ns}
5	quint	quinquaginta^{ns}	quingenti^{ns}
6	se*	sexaginta^n	sescenti^n
7	sept*	septuaginta^n	septingenti^n
8	oct	octoginta^{mx}	octingenti^{mx}
9	non*	nonaginta	nonagenti

The prefixes are attached in the order of units, tens, and then hundreds for historical reasons.

Like many prefixes, slight modifications are necessary depending on the context in which they are used. Modifications are needed for the four units marked with $\star$ above. They are as follows:

- "tre" becomes "tres" when used directly before a component marked with an s and when used directly before illion it becomes "tr" as in "trillion".
- "se" becomes "sex" when used directly before illion or a component marked with an x and becomes "ses" when used directly before a component marked with an s.
- "sept" and "non" become "septem" and "novem" when used directly before a component marked with an m and become "septen" and "noven" when used directly before a component marked with an n.

Examples:

- quintdecisescenti is 615 so quintdecisescentillion is the 615$^{\text{th}}$ illion, 10^{1848} as a base-ten exponent.
- septemoctigenti is 807 so septemoctigentillion is the 807$^{\text{th}}$ illion, 10^{2424} as a base-ten exponent.

163. Write the name and the base-ten exponent which represent the 237$^{\text{th}}$ illion.

164. Write the name and the base-ten exponent which represents that 649$^{\text{th}}$ illion.

165. Name the numbers 10^{57}, 10^{219}, and 10^{399} which you considered in Investigation **162**.

166. Name and write as a base-ten exponent the number that is the largest number that can be written in this naming scheme.

We wanted to be able to name arbitrarily large numbers. What Wachsler, Conway, and Guy did to surpass the limit in Investigation **166** was to extend their numbering scheme in blocks of 1,000 with "illi" as a separator and "nil" representing 0.

Examples:

- millinillitrillion is the $1,000,003^{\text{rd}}$ illion.
- trecentillinillioctillinonagintillion is the $300,000,008,090^{\text{th}}$ illion.

167. What illion and what base-ten exponent has the name novemsexagintatrecentillitresquadragintasescenti?

168. What illion and what base-ten exponent has the name trestrigintillinillimicentillion?

169. Write the name and the base-ten exponent which represents the $42,903,271^{\text{st}}$ illion.

170. Name the number $10^{24,921,846}$.

So far the numbers we have named have only had 1 as a leading digit and we have been jumping from one illion to the next skipping over all the intermediary numbers. But it is easy to fill in this gap if we simply use our everyday knowledge of naming numbers as illustrated in Investigations **138-141**.

171. Name the number $27,\underbrace{000,\ldots,000}_{36}$.

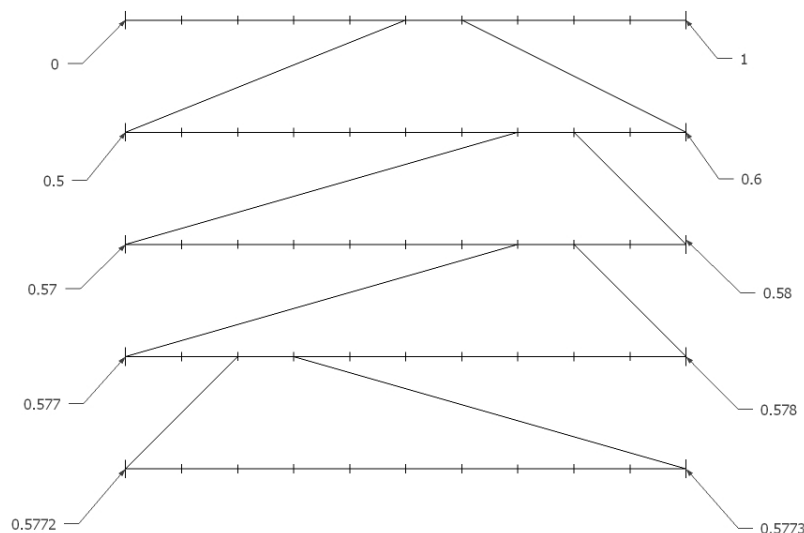**172.** Name the number $400,\underbrace{000,\ldots,000}_{150}$.

FIGURE 1. Magnifying part of the real number line.

173. Name the number $56, 320, \underbrace{000, \ldots, 000}_{318,174,639}$.

Essay 1 I'm thinking that it would be nice to have a brief essay question here that encouraged students to reflect on what was just learned. But I am not thinking of a good one. So, for this question: a) Think up a brief essay prompt related to the work above, and, b) Answer your question.

2. 0.999999... and 1

The set of *real numbers* contains all of the numbers that we work with in ordinary life: $3; 271; 1.5;$ $199.99; 1,000,000, \pi, e$, etc. One way to think of the real numbers is as what are necessary to measure lengths. For example, π is the length of the perimeter (aka the *circumference*) of a circle of radius $r = \frac{1}{2}$.

In everyday usage we generally represent real numbers using the base-ten system considered above. Above we only utilized whole numbers, here we will use decimal digits as well.

So what do decimal digits tell us? One way to think of them is as an **address** of where a given number lies on a number line. Illustrated in Figure 1 is what one would see if one repeatedly magnified a portion of the number line, with the location of several real numbers labelled.

174. Label each of the division marks in the original interval $[0, 1]$ in Figure 1.

175. Label each of the division marks in the first magnified interval $[0.5, 0.6]$ in Figure 1.

176. Why are each of the intervals divided into ten equal subintervals?

177. If you are given the decimal representation of a real number, what does each individual digit tell you about its location in the appropriately subdivided interval? Explain.

178. The magnifications in Figure 1 help us begin to locate the important **Euler-Mascheroni constant**,[3] whose decimal expansion begins 0.577215664901532, on the number line. Draw a figure which continues the illustration in Figure 1 through six more magnifications.

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[3]It is interesting to note that this important constant has been approximated to billions of decimal digits but we have no idea whether this number can be represents a fractional, *irrational*, *algebraic*, or *transcendental number*.

Here and below when we write $0.999999\ldots$ we mean the infinitely repeating decimal all of whose digits are 9. Sometimes this number is written compactly as $0.\overline{9}$. Because we will be doing arithmetic and algebra with this number we find it more useful to use the notation with the **ellipsis** $\ldots$

179. Illustrate the location of $0.999999\ldots$ as you did above for the Euler-Mascheroni constant. Use four or five magnifications. How hard would it be to continue magnifying?

180. Do you believe that $0.999999\ldots$ precisely represents a definitive, fixed, specific real number? Explain.

181. Classroom Discussion: How does $0.9999999\ldots$ compare with the number 1?

182. Use long division to precisely write $\frac{1}{3}$ as a (possibly infinite) decimal. Express your result as an equation: $\frac{1}{3} = - - - - -$.

183. Multiply both sides of your equation from Investigation **182** by 3. What does this suggest about the value of $0.999999\ldots$? Surprised?

People often object to the result in Investigation **183** because $0.999999\ldots$ and 1 appear so different. But remember, the two expressions $0.999999\ldots$ and 1 are simply symbolic representations of real numbers. And there many representations of numbers that are not unique. For example, we can write the real number 3 as $\frac{6}{2}$, $\frac{21}{7}$, $\sqrt{9}$, III, $3.\overline{0}$, or even 11_2, the base two notation that all computers use to represent the number 3. (Link to trick above.)

184. Give several real-life examples of objects that we commonly represent in different ways.

185. In thinking about $0.999999\ldots$ as a representation of a number we might know more readily in a different symbolic guise, let us use algebra to help us. Since we aren't sure of the identity of $0.999999\ldots$, let's set $x = 0.999999\ldots$ Determine an equation for $10x$ as a decimal.

186. Using your equation for $10x$ in the previous investigation, complete the following subtraction:

$$10x =$$
$$-x = 0.9999999\ldots$$
$$\rule{3cm}{0.4pt}$$
$$=$$

187. Solve the resulting equation in Investigation **186** for x. Surprised?

Seventh Grader Makes Amazing Discovery

New discoveries and solutions to open questions in mathematics are not always made by professional mathematicians. Throughout history mathematics has also progressed in important ways by the work of "amateurs." Our discussion of $0.9999999\ldots$ provides a perfect opportunity to see one of these examples.

As a seventh grader **Anna Mills** (American Writer and English Teacher; 1975 -) was encouraged to make discoveries like you have above about the number $0.999999\ldots$ Afterwards Anna began experimenting with related numbers on her own. When she considered the (infinitely) large number $\ldots999999.0$ she was surprised when her analysis "proved" that $\ldots999999.0 = -1$! She even checked that this was "true" by showing that this number $\ldots999999.0$ "solves" the algebraic equations $x + 1 = 0$ and $2x = x - 1$, just like the number -1 does.

Encouraged by her teacher and her father to pursue this matter, Anna contacted **Paul Fjelstad** (American Mathematician; 1929 -). Fjelstad was able to determine that Anna's seemingly absurd discovery that $\ldots999999.0 = -1$ is, in fact, true as long as one thinks of these numbers in the settings of *modular arithmetic* and *p-adic numbers*.

You can see more about this discovery in <u>Discovering the Art of Mathematics - The Infinite</u> or in Fjelstad's paper "The repeating integer paradox" in *The College Mathematics Journal*, vol. 26, no. 1, January 1995, pp. 11-15.

Here's an alternative way to think about the relationship between $0.999999\ldots$ and 1, one based on the theory of *limits* that underlies the almost universally accepted framework for the system of real numbers that have been precisely defined by mathematicians.

188. Evaluate the truth of the following claim:

> Unless they are equal, any two real numbers have a fixed, non-zero distance that separates them.

Explain.

189. Classroom Discussion: Is there some fixed, non-zero distance between the real numbers $0.999999\ldots$ and 1?

The discussion question Investigation **189** is a yes or no question - these are the only possible answers.[4]

Let us start our investigations by assuming that the answer to our question is "yes", there is some fixed, non-zero distance between $0.999999\ldots$ and 1.

190. Write down a *really, really small* fixed, specific non-zero number to estimate the hypothetical distance between $0.999999\ldots$ and 1. Denote this distance by the Greek letter epsilon[5] which is written as ϵ.

191. Which number is closer to 1, 0.9 or $0.999999\ldots$? Explain.

192. What is the distance between 1 and 0.9?

193. Which number is closer to 1, 0.99 or $0.999999\ldots$? Explain

194. What is the distance between 1 and 0.99?

195. Which number is closer to 1, 0.999 or $0.999999\ldots$? Explain

196. What is the distance between 1 and 0.999?

197. You should see a pattern forming. Describe this pattern precisely.

198. By adding enough zeroes, you can find a number of the form $0.00\ldots01$ that is smaller than the ϵ you chose in Investigation **190**. Do so explicitly.

199. Use Investigation **198** and the pattern you described in Investigation **197** to find a number of the form $0.99\ldots9$ so that the distance from $0.99\ldots9$ to 1 is less than the number you found in Investigation **198**.

200. You should now be able to conclude that the distance from $0.999999\ldots$ to 1 is less than ϵ, contradicting your choice of ϵ in Investigation **190**. Explain.

It didn't matter how small the ϵ you chose in Investigation **190** was, this process can be repeated.

201. Explain why these investigations show that there cannot be any fixed, non-zero distance between $0.9999999\ldots$ and 1.

202. Explain why this proves that $0.999999\ldots = 1$ as real numbers.

This type of argument is a fairly modern one, due in large part to the work of **Augustin-Louis Cauchy** (French Mathematician; 1789 - 1857). His definition of limits in this way was the culmination

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[4]Technically this assumption is known as the *Law of the Excluded Middle*. While there are some mathematical philosophies and systems of logic that do not include the Law of the Excluded Middle as an axiom, this law is generally accepted and we will use it freely here.

[5]This is the typical notation for this type of limit argument. Because it is so used, the great 20th century mathematician **Paul Erdös** (Hungarian Mathematician; 1913 - 1996), who is quoted and referenced in many books in this series, used to call children "epsilons".

of a period of great crisis in mathematics during the middle of the nineteenth century. This crisis was foretold as early as the advent of calculus when **Bishop George Berkeley** (Irish Philosopher and Theologian; 1685 - 1753) wrote about the "infidel mathematicians" and their use of "infinitesimals", saying:

> And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?

Much more about these issues are included in the companion book Discovering the Art of Mathematics - Calculus in this series.

We close our comments by noting that there are different systems of numbers than the real numbers. In particular, the *surreal numbers* considered in the companion book Discovering the Art of Mathematics - The Infinite are a system of numbers that include infinitely many different infinitely small non-zero numbers. And this opens Pandora's Box right back up.

203. Have these investigations changed your answer to Investigation **181**? Explain.

3. Understanding Exponents Using Patterns

[Need a context. Can talk about the use of zero, negative numbers, etc. This is already in the infinite thing. How to cross-reference it here? It is important because it illustrates nicely the development of thought in mathematics. What is obvious to one generation is not to another. The great quote by Schrodinger about root 2. Sand Reckoner is a good segue. It is really just shorthand. But it becomes quite powerful and is now used in remarkably sophisticated ways. (Set the stage for Euler's formula at end.) The use of exponents in algebra is often a problematic area because one must extend fairly intuitive conventions into more abstract realms. Here we illustrate how patterns naturally explain that extension.]

204. Using the definition of powers above, describe what 2^4, 5^2 and 7^3 mean. Then convert each of these into base-ten numbers with no exponents.

205. Complete the following table by filling in five more rows with different positive integer values of m and n:

m	2^m	2^m as base-ten #	n	2^n	2^n as base-ten #	$2^n \times 2^m$ as base-ten #	Is $2^n \times 2^m$ a power of 2?
3	2^3	8	5	2^5	32	256	Yes. $256 = 2^8$.

206. Based on your table, for positive integer values of m and n is $2^n \times 2^m$ always a power of 2?

207. If you answered Investigation **206** in the affirmative, find a formula which expresses $2^n \times 2^m$ as a single power of 2.

208. Return to the definition of powers and show why the result in Investigation **207** really follows from the definition of powers. (I.e. provide a proof of the result in Investigation **207**.)

209. Repeat Investigations **205-208** for a positive integer base different than $a = 2$.

210. Will the rules for exponents you found in Investigation **207** and Investigation **209** hold for any base a ¿ 0? Explain why and how you know this.

Having determined patterns in the multiplication of powers to a common base, a natural question is whether there is a corresponding rule for division.

211. Complete the following table by filling in five more rows with different positive integer values of m and n:

m	2^m	2^m as base-ten #	n	2^n	2^n as base-ten #	$2^n \div 2^m$ as base-ten #	Is $2^n \div 2^m$ a power of 2?
3	2^3	8	5	2^5	32	4	Yes. $4 = 2^2$.

(Warning: There are some issues for $m \geq n$ that will have to be considered later.)

212. Based on your table, for positive integer values of m and n is $2^n \div 2^m$ always a power of 2 for $m < n$?

213. If you answered Investigation **212** in the affirmative, find a formula which expresses $2^n \div 2^m$ as a single power of 2 whenever $m < n$.

214. Return to the definition of powers and show why the result in Investigation **??** really follows from the definition of powers. (I.e. provide a proof of Investigation **??**.)

215. Repeat Investigations **211-214** for a positive integer base different than $a = 2$.

216. Will the rules for exponents you found in Investigation **??** and Investigation **215** hold for any base $a > 0$? Explain why and how you know this.

217. The type of patterns and reasoning you used for $2^n \times 2^m$ and $2^n \div 2^m$ can naturally be extended to provide an analogous result for $(2^n)^m$ where m and n are positive integers. Find such a result and explain how you know it is valid.

The results you have found in Investigations Investigation **207**, Investigation **??**, and Investigation **217** are the classical "Rules for Exponents" that most of us were exposed to in Middle School.

218. Without wondering whether it is valid or not, apply your division rule from Investigation **??** to each of the expressions $2^5 \div 2^5, 2^3 \div 2^5$, and $2^4 \div 2^7$.

In Investigation **218** the division rule gave rise to an exponent which are 0 or even negative numbers. The definition of 2^4 should seem like second nature, we're used to "2 times 2 times 2 times 2." But 2^{-3}? You certainly "can't have a times itself a negative number of times."

Exponents and a suitable notation to express them is a human construct. It is part of a language - the language of algebra. Intuitive ideas and numerical patterns give rise to precise definitions. Yet this whole process would be of small value if the use of exponents in mathematics was limited to the narrow cases considered above.

Like any other language, mathematics grows to accommodate new needs. Here we look to extend the notion of exponents to include 0 and negative numbers. How do we do this? Patterns.

219. Complete the table that is begun below:

Row	Power Notation	Definition	Numerical Value
5	2^5	$2 \times 2 \times 2 \times 2 \times 2$	
4			16
3	2^3	$2 \times 2 \times 2$	
2		4	
1	2^1	2	

220. As you move from Row 1 of the table to Row 2 of the table, describe what happens to the entries in each column.

221. What happens to the entries in each of the columns as you move from Row 2 to Row 3? Row 3 to Row 4? Is there a pattern?

222. Now describe what happens to the entries in each column as you move from Row 5 to Row 4.

223. What happens to the entries in each of the columns as you move from Row 4 to Row 3? Row 3 to Row 2? Is there a pattern?

224. Following the pattern in Investigation **223**, you should be able to extend the table down another row - albeit leaving the definition column blank since we have no formal definition (yet) and no intuitive idea what should appear there.

225. Repeat Investigation **224** to extend the table to have five more rows, building meanings for the powers $2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}$, and 2^{-5}.

226. If they aren't already, convert the numerical values in each of the bottom five rows to fractions with numerators 1 and denominators a power of 2. Use this to provide a definition for powers with negative exponents:

Definition For bases $a > 0$ and exponents $m > 0$, define $a^{-m} = \frac{1}{?}$.

227. With this new definition, illustrate how your division rule for exponent works when $m \geq n$ as well.

Mathematicians have been able to extend our basic intuitive understanding of exponents to deal with essentially arbitrary bases and exponents. The most remarkabe illustration of how far we can extend the basic notion of exponents is certainly **Euler's formula**, discovered by **Leonhard Euler** (; -) around ????. The most elementary proof of the validity of this formula involves ideas from calculus relying on infinite series and periodic functions closely related to harmonics in music.

His formula is:

$$e^{i \cdot \pi} + 1 = 0.$$

Here π is the ubiquitous numerical constant related to circles and spheres, a transcendental irrational number which is approximated by the base-ten decimal $3.1415\ldots$ e is another fundamental mathematical constant, named after Euler himself, which arises in fundamental growth problems in biology, economics, and many other areas. It is also a transcendental irrational number which is approximated by the base-ten decimal $2.718\ldots$ i is the imaginary unit $i = \sqrt{-1}$ which gives rise to the complex numbers when combined with the real numbers. This number i is a square root we have long been told by our high school teachers does not exist despite the fact that "There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another." [6]

228. We've just noted that π, e, and i are fundamentally important numbers. What about 0 and 1? Are there other fundamentally important numbers that cannot be made from these five numbers? Explain.

229. What four fundamental mathematical operations are involved in Euler's formula? Are there fundamental operations that are not part of this formula? Explain.

230. In light of your answers to Investigation **228** and Investigation **229**, how remarkable is it that these five numbers and these four mathematical operations are expressed so concisely by this one formula? Explain, perhaps by attempting to create a simpler analogue or comparing with some other unifying statement from some other area of intellectual thought.

In regard to a similarly curious formula, $i^i = \frac{1}{\sqrt{e^\pi}}$, **Benjamin Pierce** (American Mathematician; 1809 - 1880), the "Father of American mathematics", said, "We have not the slightest idea of what this equation means, but we may be sure that it means something very important."

4. Concluding Reflections

Albert Einstein (; -) remarked, "It is not so very important for a person to learn facts. For that he does not really need a college. He can learn them from books. The value of an education in a liberal arts college is not the learning of many facts but the training of the mind to think something that cannot be learned from textbooks."

Essay 2: Compare and contrast the approach above for learning about exponents to that which you experienced in middle and/or high school. Relate this comparison/contrast to Einstein's quote, providing either supporting evidence for or dissenting views against Einstein's claim.

Noted mathematical author **Ian Stewart** (; -) once noted, "One of the biggest problems of mathematics is to explain to everyone else what it is all about. The technical trappings of the subject, its symbolism and formality, its baffling terminology, its apparent delight in lengthy calculations: these tend to obscure its real nature. A musician would be horrified if his art were to be summed up as 'a lot of tadpoles drawn on a row of lines'; but that's all that the untrained eye can see in a page of sheet music."

Essay 3: In this chapter we have described a few of the ways that common notions in mathematics are extended far beyond their original intuitive meaning. In this sense, is the language of the mathematician that much different in its history, development and accessibility much different than spoken languages? Much different from other formalized languages such as musical notation? Explain.

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[6]Keith Devlin, from <u>Mathematics: The New Golden Age</u>.

CHAPTER 6

Existence of $\sqrt{-1}$

[Complex numbers are] a fine and wonderful refuge of the divine spirit - almost an amphibian between being and non-being.[1]

G. W. Leibniz (German mathematician; -)

I have often been surprized, that Mathematics, the quintessence of Truth, should have found admirers so few and so languid. Frequent consideration and minute scrutiny have at length unravelled the cause, viz. that though Reason is feasted, Imagination is starved; whilst Reason is luxuriating in it's proper Paradise, Imagination is wearily travelling on a dreary desert.

Samuel Taylor Coleridge (English poet and philosopher; 1772 - 1834)

Samuel Taylor Coleridge coined the term "suspension of disbelief" which he thought was central to an audience's ability to imagine the often illusory settings of poetry, theatre and literature. In an effort to enrich mathematical imagination, try to suspend disbelief while considering the **imaginary unit**

$$i = \sqrt{-1}.$$

1. Can you explain why the product of a positive number and a negative number is a negative number? Are you confident in your explanation? Does this result make intuitive sense to you? Explain.

2. Can you explain why the product of two negative numbers is a positive number? Are you confident in your explanation? Does this result make intuitive sense to you? Explain.

3. Why are square roots of negative numbers supposedly nonexistent?

Numbers of the form $a + ib = a + \sqrt{-1}b$, with a and b real numbers, are called **complex numbers**. In $a + ib$ we call a the **real part** of $a + ib$ and we call b the **imaginary part** of $a + ib$.

4. Explain why $i^2 = -1$.

5. Simplify $i^3, i^4, i^5, \ldots$ so each is written in the standard form of a complex number. (I.e. no power of i other than i^1.)

6. Use Investigation **5** to determine a simple rule that which can be used to simplify i^n.

Does your sense of wonder go only as far as your eyes can see? Friends, do not fear what you cannot see. Mathematics, reason and imagination will reveal the truth.[2]

Arthur Square (; -)

In each of the problems below, use the standard rules of arithmetic together with your rules for powers of i to compute the indicated product, simplifying so each product is expressed in the standard form of a complex number:

7. $(2 + i) \cdot (1 + 3i)$.

8. $(3 + 4i) \cdot (1 + i)$.

9. $(2 + 3i) \cdot (-2 + 2i)$.

10. $(-2 - 2i) \cdot (-1 + 3i)$.

11. $(-1 - 3i) \cdot (2 - 2i)$.

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[1]Quoted in "Thinking the Unthinkable: The Story of Complex Numbers (with a Moral)," by Israel Kleiner, *Mathematics Teacher*, Oct. 1988.

[2]From Flatland the Movie.

12. $(-3 + i) \cdot (-2 - 2i)$.

13. $(-2 - 2i) \cdot (-4 - i)$.

14. $(4 - i) \cdot (-2 + 2i)$.

Complex numbers can be plotted on an **Argand plane** where the horizontal axis is the real coordinate and the vertical axis is the imaginary coordinate.

15. Plot each of the factors as well as the product in Investigation **7** on an Argand plane constructed on centimeter paper.

16. Repeat Investigation **15** for the data in Investigation **8** on a separate Argand plane.

17. Repeat Investigation **15** for the data in Investigation **9** on a separate Argand plane.

18. Repeat Investigation **15** for the data in Investigation **10** on a separate Argand plane.

19. Repeat Investigation **15** for the data in Investigation **11** on a separate Argand plane.

20. Repeat Investigation **15** for the data in Investigation **12** on a separate Argand plane.

21. Repeat Investigation **15** for the data in Investigation **13** on a separate Argand plane.

22. Repeat Investigation **15** for the data in Investigation **14** on a separate Argand plane.

There is nothing mysterious about points in the plane, is there? Does the existence of $(2, 1)$ concern you?

The number $2 + i$ is really no more mysterious than the point $(2, 1)$ in the *Cartesian plane*, only we have equipped the complex numbers with a multiplication for which the points in the plane generally lack. Then why does $2 + i$ concern you?

In fact, **John Stillwell** (; -) describes algebraic results of **Diophantus** (; -) which give a picture that is "extraordinarily close to what we now regard as the 'right' way to interpret complex numbers.[3]" This was nearly 2,000 years ago!

Translating the complex numbers into a different representation will provide a first sense of their utility. Just as the number $1\frac{1}{2}$ can be equally well expressed as a single fraction $\frac{3}{2}$ or a decimal 1.5, complex numbers can be written in **polar form** $r \cdot e^{i\theta}$ where $o \leq r$, $0 \leq \theta < 360$ and $e = 2.718\ldots$ is the *base of the natural logarithm* named in honor of **Leonhard Euler** (Swiss mathematician; -). r is called the **magnitude** of the complex number and is simply its distance from the origin. θ is called the **argument** of the complex number and is simply the angle between the positive real axis and the line from the origin to the complex number in question, measured in the counter-clockwise sense. The argument is measured in *radians*, but for our purposes here the translation will be successful if we use angle measures in degrees.

Example 1 i is one cm. from the origin and it's argument is $90°$. So we write $i = 1 \cdot e^{i \cdot 90°}$.

Example 2 $3 - 4i$ is 5 cm. from the origin and its argument is about $307°$. So we write $3 - 4i \approx 5 \cdot e^{i \cdot 307°}$.

Please notice we have not said what the precise role of the symbols e^i, for now they are simply part of the notational format of the way we express the polar form of the complex numbers.

23. Using a protractor and cm. ruler, approximate the polar form of each of the complex numbers in Investigation **7** - Investigation **14** and record the results in a table like the one below:

$a + ib$	Mag	Arg	Polar Form	$c + id$	...	$(a + ib) \cdot (c + id)$	...
$2 + i$				$1 + 3i$	...	$(2 + i) \cdot (1 + 3i)$	...

24. How is the argument of the product related to the arguments of the factors?

25. How is the magnitude of the product related to the magnitudes of the factors?

26. Investigations Investigation **24** and Investigation **25** should give you a very geometric characterization of the multiplication of two complex numbers. Describe it completely.

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[3]*Yearning for the Impossible: The Surprising Truths of Mathematics*, pp. 34-6. Paul Nahim has a slightly different view in the Introduction to An Imaginary Tale: The Story of $\sqrt{-1}$.
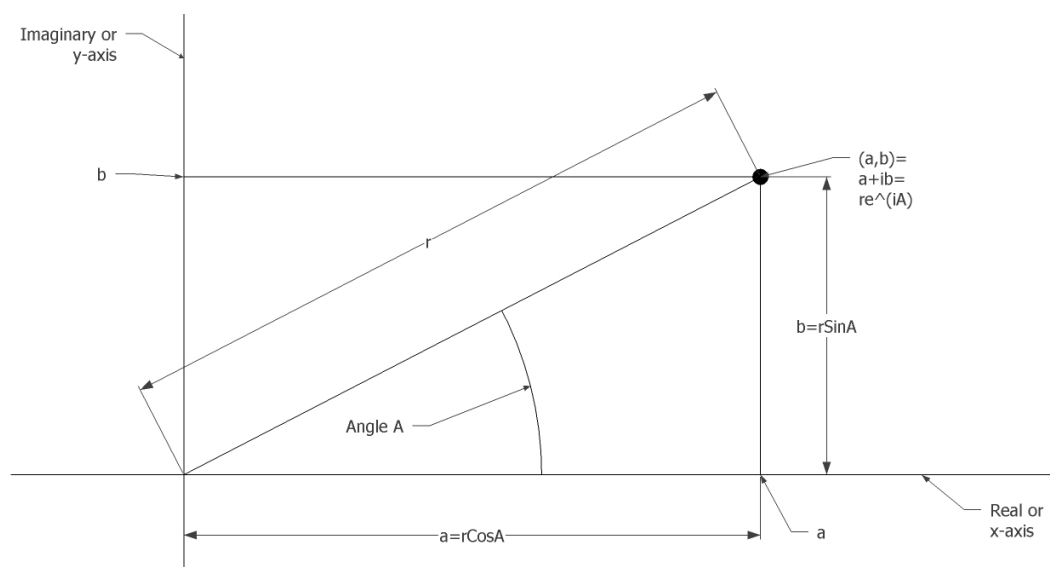
FIGURE 1. A point on the (Argand) plane.

27. Using this geometric conceptualization of multiplication, explain why the product of two positive real numbers is positive.

28. Now use this geometric view to explain why the product of a negative real number and a positive real number is negative.

29. Now use this geometric view to explain why the product of two negative real numbers is positive.

30. In the novella (resp. movie) <u>Flatland</u> by Edwin Abbott Abbott (resp. Flatland the Movie) the characters are flat geometric figures confined to a flat world, entirely unaware of a third dimension. In a dream sequence A Square (resp. Arthur Square) visits Lineland where the inhabitants are short line segments whose movement is restricted to an infinite line. The King of Lineland is only aware of his two neighbors (resp. Queens) which adjacent to him on his number-line world. This is all he knows. So he is badly startled when, from out of nowhere, his visitor appears in his world. Does this storyline resonate with your previous view of real multiplication, considered at the outset of this chapter, and the view that you now have? Explain.

> It was a wild thought, in the judgement of many; and I too was for a long time of the same opinion. The whole matter seemed to rest on sophistry rather than on truth. Yet I sought so long, until I actually proved this to be the case.[4]
>
> **Enrico Bombelli** (; -)
>
> Because all conceivable numbers are either greater than zero, less than zero or equal to zero, then it is clear that the square root of negative numbers cannot be included among the possible numbers And this circumstance leads us to the concept of such numbers, which by their nature are impossible and ordinarily are called imaginary or fancied numbers, because they exist only in the imagination.[5]
>
> **Leonhard Euler** (Swiss mathematician; -)

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[4]Quoted in "Thinking the Unthinkable: The Story of Complex Numbers (with a Moral)" by Israel Kleiner, Mathematics Teacher, October 1988, pp. 583-92.

[5]Quoted in "Thinking the Unthinkable: The Story of Complex Numbers (with a Moral)" by Israel Kleiner, Mathematics Teacher, October 1988, pp. 583-92.

31. If $\sqrt{-1}$ "exists in our imaginations" as Euler claims, then does it really exist? Is this existence the same or different that numbers like $0, -1, \sqrt{2}$ or π? Explain.

32. Return to the quote by Cooleridge that opens this chapter. Why does it appear in this chapter? Do you think that the mathematics you have investigated here may refute Cooleridge's claim?

1. Expanding Our Notion of Numbers via Solving Equations

No justification was given for the introduction of imaginary and complex numbers above. Here we trace a bit of the evolution of number through the solution of equations - a historically important timeline.

In the early sixteenth century, mathematical problem solving competitions were common. Scholarly reputations were largely based on these contests becaues "not only could an immediate monetary prize be gained by proposing problems beyond the reach of one's rival, but the outcomes of these challenges strongly influenced academic appointments.[6] One of mathematics' great disputes arose out of these competitions.

The players in the dispute were three. **Antonio Maria Fiore** (; -) was the pupil of **Scipione del Ferro** (Italian mathematician; 1465 - 1526) who was one of the greatest of the early competitors. **Nicolo Tartaglia** (Italian mathematician; 1500 - 1557) saw his father killed and had his face nearly destroyed in the sack of Brescia as a 12 year old. Despite living in poverty, Tartaglia "was determined to educate himself.[7] Throughout his life he was known as the "stammerer" because of the difficulties to his speaking that his childhood injuries caused. **Girolamo Cardano** (Italian mathematician; 1501 - 1576), aka Cardan, had a life that was "deplorable." He "divided his time between intensive study and extensive debauchery.[8]"

Cardano horned in on the ongoing competitions between Fiore and Tartaglia, pretending to befriend Tartaglia. Once he gained his confidence Cardano managed to have Tartaglia share his secret to solving an important class of cubic equations. Cardano then published the results in his famous book Ars Magna in 1545, beginning "one of the bitterest fueds in the history of science, carried on with name-calling and mudslinging of the lowest order.[9]"

In 1572 **Enrico Bombelli** (; -) used a leap of imagination to compute with $\sqrt{-1}$. He did this when considering the equation $x^3 = 15x + 4$ for which Cardano's cubic equation predicts a root of $\sqrt[3]{2 + 11 \cdot \sqrt{-1}} + \sqrt[3]{2 - 11 \cdot \sqrt{-1}}$.

33. Show that $\sqrt[3]{2 + 11 \cdot \sqrt{-1}} = 2 + i$.

34. Show that $\sqrt[3]{2 - 11 \cdot \sqrt{-1}} = 2 - i$.

35. Use these results to find the root predicted by Cardano's formula.

36. Show that this (*real*) number is a root of the cubic.

While this root could have been found in other ways, we see Cardano's formula is that much more powerful with the use of complex numbers as its use is now not limited. The same can be said for the solution of quadratic equations. You need not worry about the dreaded ***discriminant*** $b^2 - 4ac$ being nonnegative. So we have the first known illustration of a famous maxim of **Jacques Hadamard** (; -):

> The shortest and best way between two truths of the real domain often passes through the imaginary one.[10]

37. Solve the equation $2x - 6 = 0$.

38. Solve the equation $6x - 2 = 0$.

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[6]From The History of Mathematics: An Introduction by David M. Burton, p. 305.

[7]Ibid, p. 306.

[8]Ibid, p. 307.

[9]Ibid, p. 308.

[10]From "An Essay on the Psychology of Invention in the Mathematical Field."

39. If you did not know anything about fractions, what would you say about the soluability of the equation in Investigation **38**?

40. There is nothing particularly problematic about the solution to the equation in Investigation **38**, unless... Express the solution as a decimal. Is there anything troubling or potentially problematic about this number?

41. Solve the equation $2x + 6 = 0$.

42. Find out when negative numbers first came into regular use. Without negative numbers, what would you say about the soluability of the equation in Investigation **41**?

43. Your solution in Investigation **41**, can you show me this quantity concretely? I.e. does it exist in our physical reality? Where?

44. Solve the equation $x^2 - 2 = 0$.

45. Can you describe your solutions to the equation in Investigation **44** exactly? I.e. what is their exact numerical value?

46. Like $\sqrt{9} = 3$, some square roots are simple. However, whenever n is a positive integer and $\sqrt{n}$ is not a whole number then the decimal explansion of $\sqrt{n}$ is an infinite decimal that *never* repeats. So, would you say that your solution to Investigation **44** is an exact, conrete entity which exists in our physical world?

> The idea of the continuum seems simple to us. We have somehow lost sight of the difficulties it implies...We are told such a number as square root of 2 worried Pythagoras and his school almost to exhaustion. Being used to such queer numbers from early childhood, we must be careful not to form a low idea of the mathematical intuition of these ancient sages, their worry was highly credible.
>
> **Erwin Schrödinger** (German physicist; -)

Let us consider the equation $y = x^2 + 1$ where x and y are real numbers.

47. Graph this equation in the ***Cartesian plane***, the standard $x - y$ plane that you were likely introduced to in middle or high school.

48. As we have used them, how does the Argand plane differ from the Cartesian plane?

49. What does this graph suggest about real solutions to the equation $x^2 + 1 = 0$?

50. Suppose now that we considered the equation $w = z^2 + 1$ where z and w are complex numbers. In how many dimension would the graph of this "simple" quadratic function live? Explain.

51. Find the two distinct complex solutions to the equation $z^2 + 1 = 0$.

52. Based on what you have seen above, do you think that these numbers are any less legitimate than the solutions to the other equations considered above? Explain.

What about higher order equations, can we find new solutions there using complex numbers? Let's begin by investigating higher powers of complex numbers.

53. Using graph paper, draw an Argand plane. Choose a point z with magnitude greater than one and relatively small argument. Graph z on your plane.

54. Graph the points $z^2, z^3, z^4, ..., z^{10}$ on your plane as well.

55. With line segments or a curve of your choice, connect z to z^2, z^2 to z^3, z^3 to z^4, etc. What shape do you see?

56. Will something similar happen no matter what point z you start with? (Hint: Think about different categories of magnitudes and arguments you might choose.)

57. Find a dozen spirals in nature.

58. The world "natural" has come up many times in our discussion of the complex numbers. Do you find them more natural now?

59. Using your experience with spirals above, find the three distinct complex solutions to the equation $z^3 - 1 = 0$.

60. Find the three distinct complex solutions to the equation $z^3 + 8 = 0$.

61. For each of the equations whose solutions are considered in this chapter, record the **_degree_** (the highest exponent of the variable x), the number of real solutions and the number of complex solutions (which includes the real solutions as well). Notice something?

62. The **Fundamental Theorem of Arithmetic** says that every polynomial of degree n (i.e. which has the form $a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$ has exactly n complex roots regardless of whether the coefficients a_j are complex are real. What does this suggest about the most natural arena to solve polynomial equations?

Here's one last example if you still question whether the complex numbers are very natural.

Above we measured the arguments of complex numbers in degrees. Typically **_radian_** meausre for angles should be used. In radians $2\pi = 360°$, so $\frac{\pi}{2} = 90°$, etc.

63. What is the degree equivalent of the radian measure π?

64. Determine what complex number $e^{i \cdot \pi}$ represents.

65. Rewrite the equation above so all nonzero terms are on the left.

66. What important numbers does your equation include? What important operations? Is it remarkable that one valid equation contains so many fundamental mathematical objects and operations?

2. Further Investigations

2.1. Trigonometric Identities. In his wonderful book Visual Complex Analysis, **Tristan Needham** (; -) tells us:

> All trigonometric identities may be viewed as arising from the rule for complex multiplication... Every complex equation says two things at once.

Is this really the case that the entire zoo of trigonometric identites are encoded in complex multiplication? Let's give the following a try:

$$cos(\theta + \phi) = cos\theta cos\phi - sin\theta sin\phi.$$

67. Write $e^{i \cdot (\theta + \phi)}$ in the standard form of a complex number.

68. Explain why $e^{i \cdot (\theta + \phi)} = e^{i\theta} \cdot e^{i\phi}$.

69. Write both of the polar forms in Investigation **68** as complex numbers in the standard form.

70. Compute the product to express the right hand side of the equation in Investigation **68** in standard form.

71. Explain why the real parts of the expressions in Investigation **68** and Investigation **67** can be equated to derive the desired formula.

72. What happens when you equate the imaginary parts?

73. How hard was it to derive these formulas this way?

2.2. Legitimizing the Formula $e^{a+ib} = e^x(cosy + isiny)$**.** Above the polar representation was introduced as a notation. So the remarkable formula $e^{i \cdot \pi}$ might seem spurious. Here you see how this formula is justified.

Begin by defining $e^{i \cdot y} = cosy + i \cdot siny$ for y real.

74. Use the rules for exponents to find a formula for e^{x+iy} in the form of a standard complex number.

75. Use your formula in Investigation **74** to show that several of polar forms that you determined above agree with your previous results.

Still, "defining" $e^{i \cdot y}$ as we have might seem questionable. Here you show how this definition is perfectly natural.

76. Use a graphing calculator or online grapher to graph the functions $siny$ and $y - \frac{y^3}{3 \cdot 2}$ on the same graph.

77. Repeat Investigation **76** for the functions $siny$ and $y - \frac{y^3}{3 \cdot 2} + \frac{y^5}{5 \cdot 4 \cdot 3 \cdot 2}$.

78. Repeat Investigation **76** for the functions $siny$ and $y - \frac{y^3}{3 \cdot 2} + \frac{y^5}{5 \cdot 4 \cdot 3 \cdot 2} - \frac{y^7}{7!}$.

79. You should see a pattern in the polynomial functions you are being asked to graph together with the *sin* function. Find the next three such polynomials and graph them together with the *sin* function.

80. How do the graphs of the polynomials compare to the graph of the *sin* function?

81. As you continue to use the higher degree polynomials that are generated by this pattern (these are called the ***Taylor polynomials*** for sin what do you think will happen?

In fact, the Taylor polynomials *converge* to the *sin* function for any value of y. I.e. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

82. The ***Taylor series*** for *cosy* is $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Graph several of the Taylor polynomials together with the *cos* function to convince yourself that they too look to converge.

83. The Taylor series for e^y is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Graph several of the Taylor polynomials together with the exponential function to convince yourself that they too look to converge.

84. Substitute iy in as the variable to find a Taylor series representation for $e^{i \cdot y}$.

85. Reduce all of the powers of i in the Taylor series so no higher power than the first occurs.

86. Group the real terms together. What do you notice?

87. Group the imaginary terms together. What do you notice?

There are a few technical details that have to be cleaned up to have a rigorous proof.[11] But you have what Euler and the mathematicians prior to the late nineteenth century would certainly consider a proof of the legitimacy of this important equation.

3. Connections

> There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another.[12]

Keith Devlin (; -)

Here we provide references where interested readers can find out about some of the important, "real-world" applications of complex numbers:

- Fluid Flow (hence much of aeronautical engineering) - Chapter 2, Section 6 of Complex Variables by Norman Levinson and Raymond M. Redheffer.
- Electrical Engineering - Chapter 5, Section 3 of An Imaginary Tale: The Story of $\sqrt{-1}$ by Paul J. Nahin.
- Kepler's Laws and Satelite Orbits - Chapter 5, Section 3 of An Imaginary Tale: The Story of $\sqrt{-1}$ by Paul J. Nahin.

To get a sense of the central importance of complex numbers to mathematics it is interesting to view the history of the prime number theorem. This theorem describes the distribution of the prime numbers, the building blocks for all the whole numbers under multiplication, by approximating the density of primes. Specifically, the **prime number theorem** says that as N grows without bound the density of primes in the range $1, 2, 3, \ldots N$ is approximately $\frac{1}{ln(N)}$. Symbolically, $\frac{\pi(N)}{N} \approx \frac{1}{ln(N)}$ where the function π counts the number of primes. The great **Gauss** (; -) experimented empirically and claims to have known this result at the age of 15 or 16! **Legendre** (; -) was aware of this result about the same time, the mid 1790's. However, almost 100 years would pass before this result was proven, independently, in 1896 by **Jacques Hadamard** (; -) and **Charles Jean de la Vallée-Poussin** (; -)! Both of these proofs involved complex numbers and complex function theory. In fact, all proofs through 1948 did! The most important result about the whole numbers and a "'real variable' proof of

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[11]For example, can you believe that it is not always ok to rearrange the order of terms in an infinite series? See Discovering the Art of Mathematics - The Infinite for discussion.

[12]From Mathematics: The New Golden Age

the prime number theorem, that is to say a proof not involving explicity or implicitly the notion of an analytic function of a complex variable, has never been discovered, and we can now understand why this should be so... [13]" It was not until 1949, over 150 years since its formulation, that the Prime Number Theorem was proven without the use of complex numbers and complex functions! These proofs were due to **Alte Selberg** (; -) and, perhaps not independently, **Paul Erdös** (Hungarian mathematician; -).

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[13]From the 1932 text The Distribution of Prime Numbers by Albert Ingham; cited on p. 125 of Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics by John Derbyshire.

Applications of Algebra: Tuning and Intervals

1. Fractions: How perfect is Pythagorean Tuning?

Sitting on the riverbank, Pan noticed the bed of reeds was swaying in the wind, making a mournful moaning sound, for the wind had broken the tops of some of the reeds. Pulling the reeds up, Pan cut them into pieces and bound them together to create a musical instrument, which he named "Syrinx", in memory of his lost love

Ovid (Roman Poet; 43 BC - AD 18/19)

Have you ever watched someone tune a guitar? Or maybe even a piano? The lengths of the strings have to be adjusted by hand to exactly the right sound, by making the strings tighter or looser. Bue how does the tuner know which sound is the right one? This question has been asked throughout history and different cultures at different times have found different answers. Many cultures tune their instruments differently than we do. Listen for instance to the Indian instrument *sarod* in `http://www.youtube.com/watch?v=hobK_8bIDvk`. Also, 2000 years ago, the Greek were using different tuning ideas than we do today. Of course the Greek did not have guitars or pianos at that time, but they were still thinking about tuning for the instruments they had and about the structure of music in general. The **pan flute**, one of the oldest musical instruments in the world, was used by the ancient Greeks and is still being played today. It consists of several pipes of bamboo of increasing lengths. The name is a reference to the Greek god Pan who is shown playing the flute in Figure 1.

FIGURE 1. Pan playing the pan flute.

For the following investigations you need to make your own "pan flute" out of straws. Straws for *bubble tea*[1], work better than regular straws since they have a wider diameter. You need to plug the bottom with a finger to get a clear pitch. Put your lower lip against the opening of the straw and blow across the opening (but not into it). It helps to have some tension in the lips, as if you were making the sounds "p". Also, for shorter straws you need more air pressure than for longer straws.[2]

1. Take a straw and cover the bottom hole while blowing over the top hole. Practice until you can hear a clear note. *Why* do you think we hear a sound?

2. Do you think the sound will be different if the straw is longer or shorter? Explain your thinking.

3. Take a rubber band, hold it tight between two hands and have someone pluck it. Can you hear a clear note?

4. Take a rubber band, stretch it over a container and pluck it. Can you hear a clear note? Why do we hear a sound?

5. Do you think the sound will be different if the rubber band is longer or shorter? Tighter or looser? Explain your thinking.

6. **Classroom Discussion:** How is sound generated? What exactly is vibrating? What is a *sound wave*? How do different musical instruments like drum, guitar, violin and trumpet generate sound?

For the next investigations we will use the modern piano as a reference tool, so that we can compare our sounds and give them labels. Even with the piano it is quite difficult to hear if two sounds are the same or not. If you have difficulties, turn to someone who has practiced music for a long time for support.

7. Take one straw and cut it such that it has the sound of any white key on a piano (except for the B key, see Figure 2.
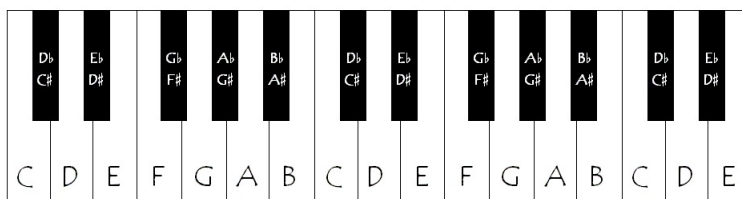

FIGURE 2. piano keys with labels.

We will discover later why the B key doesn't work.) You can go to
http://www.play-piano.org/play_online_piano_piano.html to use the online piano.

8. Take a second straw and cut it so that it has a length of $\frac{1}{2}$ of the first straw.

9. Take a third straw and cut it so that it has a length of $\frac{2}{3}$ of the first straw. Be precise!

10. Take a fourth straw and cut it so that it has a length of $\frac{3}{4}$ of the first straw. Be precise!

11. Compare the sounds of 2 straws at a time. We call two notes sounding at the same time an **Interval**. We write e.g. $(1, \frac{2}{3})$ for the interval of the first straw and the straw with length $\frac{2}{3}$. Listen carefully: which two straws sound the most alike? You can also sing the notes of the 2 straws and listen to the interval to make your decision.

12. **Classroom Discussion:** Share your intervals with the class. Decide together which fraction gives the "most alike" interval.

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[1] "Bubble tea" is the American name for pearl milk tea from Taiwan. You need straws with a larger diameter to drink bubble tea, since the tea contains small balls made of starch.

[2] Tubes with diameter $\frac{1}{10}$ of their length are easiest to play!

We call the interval that sounds the most alike an **Octave**. Human brains seem to be hard-wired to perceive these sounds as alike or even the same. The thalamus is a part in the brain of mammals that is built in layers of neurons that correspond to octaves. See Figure 3. Additionally research shows that rhesus monkeys have "same" responses to melodies that are one or two octaves apart but "different" responses to other melody shifts.

FIGURE 3. Thalamus in the Human Brain.

This explains why we can find octaves in cultures all over the world even though their music may sound very different. Even though all cultures share octaves, there are many ways to divide the octave into smaller intervals. We call those choices **scales**. In modern western culture, the major and minor scale are the most prominent scales. For example the C major scale corresponds to the white keys on a piano. Notice that on a piano you have to go up or down 8 white keys to travel an octave (starting on a white key and counting this first key as one of the 8).

You can go to `http://www.play-piano.org/play_online_piano_piano.html` to play the C-major scale. Take the intervals $(1, \frac{2}{3})$ and $(1, \frac{3}{4})$ and see if you can find the corresponding intervals on a piano.

13. Take your pair of straws for the interval $(1, \frac{1}{2})$. How many white keys are between the notes if you count the beginning and the end note as well?

14. Take your pair of straws for the interval $(1, \frac{2}{3})$. How many white keys are between the two straw-sounds if you count the beginning and the end key as well?

15. Why do we call the interval $(1, \frac{2}{3})$ a **fifth**[3]? Explain!

16. Why do we call the interval $(1, \frac{3}{4})$ a **fourth**? Explain!

You have probably heard of the mathematician and philosopher **Pythagoras of Samos** (Greek Philosopher and Mathematician; 570 BC - 495 BC), but did you know about the secret society called the **Pythagoreans**? The Pythagoreans believed that *everything* in the world could be explained using mathematics, including music. There is not much evidence about the life of Pythagoras and his disciples, see Further Investigation 3. However, they are credited with some important discoveries in mathematics. The Pythagoreans believed that all music could be explained using mathematics. They used, for instance, the musical fifths to get to all other notes in their scales as the next Investigations illustrate. The tuning they used is called **Pythagorean Tuning**.

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[3]We have to distinguish between the musical fifth (which is a specific interval between two notes), and a mathematical fifth (which is the fraction $\frac{1}{5}$.)

FIGURE 4. Medieval Woodcut showing Pythagoras.

17. Take the interval $(1, \frac{2}{3})$. Now take a third straw and cut it such that the length is $\frac{2}{3}$ of the previous $\frac{2}{3}$ straw. How much of your longest straw is your new, very short straw? Write your answer as a fraction and explain your reasoning.

18. What is the label of your new straw on the piano? Is it in the same octave as the first two straws? Can you see how to use the fraction to determine whether your new note is in the first octave or not? From now on we will call this octave (between our first two straws) our **main octave**.

19. Compare the two fractions $\frac{1}{1}$ and $\frac{1}{2}$, whose sounds lie an octave apart. Which fraction operation do we have to do to get from one to the other? Explain how to go up and down octaves using fractions.

20. By looking at *any* fraction, how can you tell whether the corresponding note will be in the main octave or not? Explain your reasoning.

21. Take the fraction from Investigation 17. How can we use it to get a new fraction corresponding to the same note in the *main* octave?

22. You just found a fraction representation of a note in your main octave that corresponds to a fifth above a fifth. Continue the pattern by taking the next fifth and so forth. If you can't hear the sound of your straw anymore, see if you can find the mathematical pattern to continue this quest in theory. You should find a list of 5 fractions.

23. Draw a number line from $\frac{1}{2}$ to 1 and label the first 5 fractions you found.

24. Look at a piano keyboard. How many steps are there in a fifth if you include the black keys?

25. We said earlier that a fifth corresponds to five white keys on the piano keyboard if you don't start from a B. Use Investigation 24 to argue why did we had to exclude the B.

26. Using investigation 24, how many fifths do we have to go up on a piano keyboard before we return to the same note (some octaves higher)?

27. Now we will use the fraction $\frac{2}{3}$ to go up by fifths. Find the fraction representation of the note in the main octave that corresponds to 12 fifths above your original note. Explain your strategies.

28. How far is the fraction from investigation 27 from 1? Did you expect this answer? Explain.

29. Does the chain of fifths ever end? Use fractions to explain your answer.

30. Use the chain of fifths to explain problems that arise with Pythagorean tuning.

31. Classroom Discussion: Does the chain of fifths end or not? Compare your result of the fraction computation with the result on the piano keyboard. How perfect is Pythagorean tuning?

2. Frequencies, Fractions and Ratios

It is common to measure the "height" of a note, also called **pitch**, with frequencies. The frequency measures how fast the sound wave vibrates. In a long straw (big number) the air vibrates more slowly (small number) and in a short straw (small number) the air vibrates faster (big number), which means the length of the straws is anti-proportional to the speed of vibration. For simplicity we will assume that the fractions for frequency are just the reciprocals of the fractions for length, i.e.

$$\text{frequency} = \frac{1}{\text{length}}.$$

For example a straw of length $\frac{1}{2}$ sounds with a frequency of $\frac{2}{1}$.

The unit of frequency is hertz (Hz), named after **Heinrich Hertz** (German Physicist; 1857 - 1894). 1 Hz means that an event repeats once per second.

FIGURE 5. Heinrich Hertz.

We want to redo the above investigations thinking about frequency instead of length.

32. Write the intervals $(1, \frac{1}{2})$, $(1, \frac{2}{3})$, and $(1, \frac{3}{4})$ using frequencies instead of length.

33. By comparing the two frequencies that make our main octave, which fraction operation do we use to go up and down octaves? Explain.

34. Compute the ascending fifths as above using frequencies instead of length. Explain your strategies.

35. Draw a number line from 1 to 2. Label your first 5 frequency fractions.

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36. Since the process of taking more and more fifths results in notes that sound out of tune, the Pythagoreans used the fraction $\frac{3}{4}$ to help them. Recall the key on the piano corresponding to the fourth, i.e. to the fraction $\frac{3}{4}$. How many fifths do we use to go up on the keyboard in order to get to the same note as the fourth (ignoring octaves)?

37. Why is it more accurate to work with the fourth instead of the fifths in investigation 36?

38. Label the frequency that corresponds to the fraction $\frac{3}{4}$ on your number line.

Your main straw could have been any length in the above investigations and hence correspond to any note from a white key (excluding B, of course). For the next section we will assume that it corresponds to the note C. The mathematics works out the same if you use another note as your starting point, but it makes it easier to read if we agree on a base note.

We want to discover how the Pythagorean fifths will give us the entire C-major scale!

39. Fill in the first row in table 1. If your main straw would correspond to the note C, how do the other frequency fractions we found relate to the keys on the piano? You can use the fractions you computed in the above investigations. Just match them with the C-major scale instead of the scale from your straws.

TABLE 1. Frequency Table

Note	C	D	E	F	G	A	B	C
Frequency Fraction	$\frac{1}{1}$							$\frac{2}{1}$
Ratios between Frequency Fractions								

40. Classroom Discussion: Compare the first row in table 1. Now look at the ratios[4] between adjacent fractions on your number line. Fill in row 2 in table 1. What patterns do you notice?

You just discovered the so called ***Pythagorean Tuning*** based on C. Unfortunately there are some problems with this tuning method... you will discover some of these in the next Investigations:

41. We tried to avoid the "incorrect" last fifth, also called the ***wolf interval***, by chosing the frequency $\frac{4}{3}$ instead of the last power of $\frac{3}{2}$. Will this solve the problem or will there still be a wolf interval? Explain.

42. Your piano is tuned in Pythagorean tuning based on C. Imagine you have a melody starting with the fifth CG. Do you think the song would sound "bad" if you started playing it on a different note? Explain.

So it seems that for some melodies the piano will sound in tune while for other melodies or other starting points of you melody it might sound out of tune. Musician would say: "If I played a song that uses a different ***key*** it would sound out of tune!". This *key* is not the same as a key on a keyboard. It is an abstract term roughly describing a set of notes that a piece of music is most likely to use. You can for instance say that a song is being played in the key of "C major".

That is not what we wanted! It gets even weirder:

43. Compare the ratios for a half step and a whole step in Pythagorean tuning (table 1). What do you notice? Are two half steps really a whole step? Remember to use ratios and differences in your argument.

44. Why is Pythagorean tuning a very natural way of tuning, even though problems arise?

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[4]To find the ratio between two fractions you need to divide one fraction by the other - you compute a fraction of fractions. We will divide the larger fraction by the smaller to make it easier to compare.

3. The Roots of Equal Temperament

Since the Pythagorean tuning is not the same for all *keys*, other ways of tuning were developed over time. In the 18th century **well tempering** was used, in which compromises were made such that every *key* would sound good but slightly different. One advantage of each *key* sounding different is that the mood of a piece of music can be expressed by the choice of *key*.

Since the middle of the 19th century **equal temperament** is most commonly used. This tuning requires a new mathematical idea which you will discover in the next Investigations. We know that the frequency interval $(1, 2)$ gives us an octave. It is customary in Western Music to have 12 steps in an octave. Therefore we need to find a way to split the interval between 1 and 2 into 12 "equal" steps. Since we are dealing with ratios here, we need all the steps to have the same ratio. Look back at table 1 to see 7 steps (ratios of frequency fractions) that are not all equal.

45. Split the interval between 1 and 2 into 2 "equal" steps such that the *ratios* are the same. This means we are looking for a fraction, say x, between 1 and 2, such that the ratio of 2 and x is the same as the ratio of x and 1. What is x? Describe your strategy.

46. Compare your solution with the following problem: Split the interval between 1 and 2 such that *differences* are the same. This means we have to find a number, say y, between 1 and 2 such that the difference between y and 2 is the same as the difference between y and 1. What is y? Did you get the same answer as in the last investigation?

47. Classroom Discussion: Compare the two solutions above to get "equal size" steps in the interval $[1, 2]$. Compare your strategies. What does "equal size" mean? Compare your results. Now go back to Investigation **23** and Investigation **35** and explain why we did not see any useful spacing pattern on the number lines.

48. Split the interval between 1 and 2 into 3 steps with equal ratios. Describe your strategy.

49. Split the interval between 1 and 2 into 4 steps with equal ratios. Describe your strategy.

50. Split the interval between 1 and 2 into 5 steps with equal ratios. Describe your strategy.

51. Split the interval between 1 and 2 into 12 steps with equal ratios. Describe your strategy.

52. Summarize how to find the frequencies for the *equal temperament tuning*.

53. What are some advantages and some disadvantages of *equal temperament tuning*?

You really understand Pythagorean tuning and equal temperament tuning now, and you have traveled through many centuries of music and mathematics history. Hidden in the above mathematics is some history about numbers:

The Pythagoreans believed that *every* number could be written as a fraction. Mathematicians call these numbers **Rational Numbers**. According to legend **Hippasus of Metapontum** (Greek Philosopher; 500 BC -) was put to death by Pythagoras because he had revealed the secret of the existence of **irrational numbers**: numbers that can not be written as fractions.

It might seem easy to grasp for us now, but every time mathematicians expand their ideas of numbers it is like a small revolution. And there are more than just irrational numbers! There are for instance *complex numbers* and *imaginary numbers* and *surreal numbers*. For the latter you can read the book Discovering the Art of Mathematics: The Infinite.

54. Do you find it surprising that the Hippasus was put to death?

55. Name one irrational number. Do you know more?

4. Further Investigations

The way Greek mathematicians first encountered irrational numbers was not in music, but in geometry. You will solve their problem in the next Investigation.

F1. In a square with side length equal to 1, what is the length of the diagonal?

FIGURE 6. Hippasus of Metapontum.

F2. Find a proof of the fact that $\sqrt{2}$ is an irrational number. You can look at books or go online. Explain the proof to someone else without looking at your notes to see if you fully understand it.

F3. Read "The Ashtray: Hippasus of Metapontum (Part 3)" by ERROL MORRIS published in the New York Times Opinionator. What do we *actually* know about Hippasus?

F4. Understand how to draw graphs of waves with different frequencies, see Figure 7. How does this relate to waves of air in the straws?
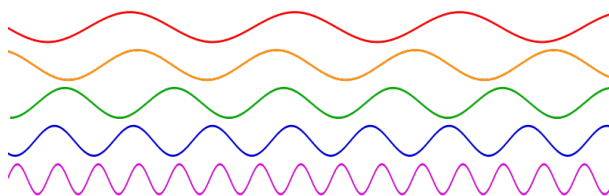

FIGURE 7. Graphs of waves with different frequencies.

Check out Ruben's Tube videos on `youtube.com`. How does this connect to graphs of sound waves? See Figure 8.

F5. In Timothy Johnson's book [**?**], you can investigate (diatonic) transposing patterns for different scales. Proving why these patterns occur is challenging and really fun.

FIGURE 8. A Ruben's Tube Experiment.